Proceedings of IJCAI-95 Morgan-Kaufmann, San Mateo, CA, 1995

Constrained and rational default logics

Artur Mikitiuk and **Mirosław Truszczyński**
Department of Computer Science, University of Kentucky,
Lexington, KY 40506-0046, USA, E-mail: {artur|mirek}@cs.engr.uky.edu

Abstract

In this paper we consider constrained and rational default logics. We provide two characterizations of constrained extensions. One of them is used to derive complexity results for decision problems involving constrained extensions. In particular, we show that the problem of membership of a formula in at least one (in all) constrained extension(s) of a default theory is Σ_2^P-complete (Π_2^P-complete). We establish the relationship between constrained and rational default logics. We prove that rational extensions determine constrained extensions and that for seminormal default theories there is a one-to-one correspondence between these objects. We also show that the definition of a constrained extension can be extended to cover the case of default theories which may contain justification-free defaults.

1 Introduction

Default logic, introduced by Reiter [1980], is one of the most extensively studied nonmonotonic systems. Several recent research monographs offer a comprehensive presentation of theoretical and practical aspects of default logic [Besnard, 1989; Brewka, 1991b; Marek and Truszczyński, 1993]. Default logic was designed to handle reasoning from incomplete information. It allows us to draw conclusions on the basis of "the lack of evidence to the contrary". This formalism assigns to a default theory a collection of theories called *extensions*. They describe possible belief sets of an agent reasoning with this theory.

All its desirable properties notwithstanding, there are situations where default logic of Reiter produces counterintuitive results. In particular, this logic does not handle well incomplete information given in the form of disjunctive clauses [Poole, 1989; Brewka, 1991a; Gelfond *et al.*, 1991; Mikitiuk and Truszczyński, 1993]. To remedy this, several modifications of default logic were proposed: disjunctive default logic [Gelfond *et al.*, 1991], cumulative default logic [Brewka, 1991a], constrained default logic [Schaub, 1992] and rational default logic [Mikitiuk and Truszczyński, 1993]. The first system introduces a new disjunction operator to handle "effective" disjunction. The latter three take into account, in one way or another, the requirement that defaults with mutually inconsistent justifications must not be used in the construction of the same extension. Not surprisingly then, they are somewhat related.

Connections between cumulative default logic and constrained default logic are studied in [Schaub, 1992]. It is shown there that these two systems are, in a certain sense, equivalent. At the same time, they are quite different from default logic of Reiter. Both commit to assumptions and have such properties as semi-monotonicity and orthogonality[1]. In addition, in each of these two logics every default theory has an extension. In the logic of Reiter all these properties hold for normal default theories but fail in the general case (in fact, for normal default theories, Reiter's default logic is essentially equivalent to constrained and cumulative default logics).

In this paper, we investigate connections between rational and constrained (and, consequently, also cumulative) default logics. Rational default logic, similarly as the logic of Reiter, lacks many of the properties of constrained default logic. In particular, default theories may have no rational extensions and rational default logic does not have the properties of semi-monotonicity and orthogonality. The reason is that rational default logic, unlike constrained default logic, does not commit to assumptions. At the same time, connections between rational and constrained default logics are quite strong. We show that every rational extension of a default theory determines a constrained extension. Moreover, we show that rational and constrained default logics coincide for the class of seminormal default theories — a much wider class of theories than normal ones, for which all four versions of default logic mentioned here are equivalent.

We also give a useful, proof-theoretic, characteriza-

[1] Schaub [1992] uses the term weak orthogonality

tion of the operator Υ, which was used in [Schaub, 1992] to define the notion of a constrained extension. Consequently, we get an equivalent definition of constrained extensions. This result allows us to design an algorithm for computing constrained extensions and to establish the complexity of reasoning with constrained extensions. Since every default theory has a constrained extension, the problem of existence of a constrained extension is, clearly, in P. We show that the problem to decide, given a formula φ, whether φ is in at least one constrained extension (in all constrained extensions) of a given default theory, is Σ_2^P-complete (Π_2^P-complete). In view of a recent result on the complexity of cumulative default logic [Gottlob and Mingyi, 1994] and our results on the complexity of rational default logic [Mikitiuk and Truszczyński, 1993], it follows that all these modes of reasoning have the same computational complexity.

Finally, let us note that Schaub did not allow justification-free defaults in his definition of constrained default logic. In this paper, we show how to extend constrained default logic to cover theories which may contain justification-free defaults.

2 Preliminaries

A *default* is any expression of the form

$$\frac{\alpha : M\beta_1, \ldots, M\beta_k}{\gamma}, \qquad (1)$$

where α, β_i, $1 \leq i \leq k$ and γ are propositional formulas. Let d be a default of the form (1). The formula α is called the *prerequisite* of d, $p(d)$ in symbols. The formulas β_i, $1 \leq i \leq k$, are called the *justifications* of d. The set of justifications is denoted by $j(d)$. Finally, the formula γ is called the *consequent* of d and is denoted $c(d)$. For a collection D of defaults by $p(D)$, $j(D)$ and $c(D)$ we denote, respectively, the sets of all prerequisites, justifications and consequents of the defaults in D. A default of the form $\frac{\alpha : M\beta}{\beta}$ ($\frac{\alpha : M(\beta \wedge \gamma)}{\gamma}$, resp.) is called *normal* (*seminormal*, resp.).

A default theory is a pair (D, W), where D is a set of defaults and W is a set of propositional formulas. A default theory (D, W) is *normal* (*seminormal*, resp.) if all defaults in D are normal (seminormal, resp.). A default theory (D, W) is *finite* if both D and W are finite.

For a set D of defaults we define

$$Mon(D) = \left\{ \frac{p(d)}{c(d)} : d \in D \right\}.$$

Given a set of inference rules A, by $Cn^A(\cdot)$ we mean the consequence operator of the formal proof system $PC{+}A$, consisting of propositional calculus and the rules in A.

In [Mikitiuk and Truszczyński, 1993] we introduced the notions of an active set of defaults and a rational extension of a default theory.

Definition 2.1 A set A of defaults is *active* with respect to sets of formulas W and S if it satisfies the following conditions:

AS1 $j(A) = \emptyset$, or $j(A) \cup S$ is consistent,

AS2 $p(A) \subseteq Cn^{Mon(A)}(W)^2$.

The set of all subsets of a set of defaults D which are active with respect to W and S will be denoted by $\mathcal{A}(D, W, S)$. □

Observe, that $\emptyset$ is active with respect to every W and S. Hence, $\mathcal{A}(D, W, S)$ is always non-empty. An application of the Kuratowski-Zorn Lemma gives the following result.

Proposition 2.1 ([Mikitiuk and Truszczyński, 1993]) *Let (D, W) be a default theory and let S be a propositional theory. Every $A \in \mathcal{A}(D, W, S)$ is contained in a maximal element of $\mathcal{A}(D, W, S)$.* □

We define $\mathcal{MA}(D, W, S)$ to be the set of all maximal elements in $\mathcal{A}(D, W, S)$.

Definition 2.2 A theory S is a *rational extension* for a default theory (D, W) if $S = Cn^{Mon(A)}(W)$ for some $A \in \mathcal{MA}(D, W, S)$. □

Schaub [1992] introduced the notion of a constrained extension by means of the following definition.

Definition 2.3 ([Schaub, 1992]) Let (D, W) be a default theory and let T be a propositional theory. Then $\Upsilon(T)$ is the pair of smallest sets of formulas (S', T') such that

CE1 $W \subseteq S' \subseteq T'$,

CE2 $S' = Cn(S')$, and $T' = Cn(T')$,

CE3 For any default $\frac{\alpha : M\beta}{\gamma} \in D$, if $\alpha \in S'$ and $T \cup \{\beta, \gamma\}$ is consistent then $\gamma \in S'$ and $\beta \wedge \gamma \in T'$.

A pair of sets of formulas (E, C) is a *constrained extension* of (D, W) if $\Upsilon(C) = (E, C)$. □

If (E, C) is a constrained extension, then we will refer to E as a *proper constrained extension*.

The following example shows that the notions of an extension, a rational extension and a proper constrained extension are different.

Example 2.1 Let us consider the default theory $(D, \emptyset)$ [Schaub, 1992], where

$$D = \left\{ \frac{:Mb}{c}, \frac{:M\neg b}{d}, \frac{:M\neg c}{e}, \frac{:M\neg d}{f} \right\}.$$

[2] In [Mikitiuk and Truszczyński, 1993] we used the classical reduct of A with respect to S, denoted A_S, instead of $Mon(A)$ both in **AS2** and in the definition of a rational extension. If A satisfies **AS1**, then $A_S = Mon(A)$. Since in this paper we will use a different notion of a reduct (Definition 3.2), we decided to reformulate the definitions of active sets and rational extensions to avoid confusion. These definitions are equivalent to the original ones.

This theory has one extension $Cn(\{c,d\})$, two rational extensions: $S_1 = Cn(\{c,f\})$, $S_2 = Cn(\{d,e\})$, and three constrained extensions: $(Cn(\{e,f\}), Cn(\{e,\neg c,f,\neg d\}))$, $(S_1, Cn(\{c,b,f,\neg d\}))$, $(S_2, Cn(\{d,\neg b,e,\neg c\}))$. $\square$

Schaub [1992] does not consider justification-free defaults, so in this paper we do not consider them either. Hence, $j(A) = \emptyset$ only if $A = \emptyset$. Moreover, in both rational and constrained default logic one can replace all justifications of a default by their conjunction. Thus, we assume that every default has exactly one justification. In Section 6 we will show how Schaub's definition can be extended to cover the case of justification-free defaults.

3 Characterizations of constrained extensions

In this section we will give two useful characterizations of constrained extensions. The first of them will be based on a proof-theoretic description of the operator $\Upsilon(T)$. We will use it in the next section to derive complexity results on reasoning with constrained default logic. The characterization requires the notion of a generating default introduced in [Schaub, 1992][3].

Definition 3.1 ([Schaub, 1992]) Let (D,W) be a default theory and S and T sets of formulas. The set of *generating defaults* for (S,T) with respect to D is defined as

$$GD_D^{(S,T)} = \left\{ \frac{\alpha : M\beta}{\gamma} \in D : \alpha \in S, T \cup \{\beta,\gamma\} \not\vdash \bot \right\}.$$

$\square$

Schaub [1992] proved the following properties of generating defaults.

Theorem 3.1 *Let (S,T) be a constrained extension of a default theory (D,W). Then*

1. $S = Cn(W \cup c(GD_D^{(S,T)}))$.

2. $T = Cn(W \cup c(GD_D^{(S,T)}) \cup j(GD_D^{(S,T)}))$.

3. *There is an enumeration $\delta_1, \delta_2, \ldots$ of the defaults in $GD_D^{(S,T)}$ such that for every $i = 1,2,\ldots$, $W \cup c(\{\delta_1,\ldots,\delta_{i-1}\}) \vdash p(\delta_i)$.* $\square$

This theorem implies the following useful corollary.

Corollary 3.2 *Let (S,T) be a constrained extension of a default theory (D,W). Then $S = Cn^{Mon(GD_D^{(S,T)})}(W)$.* $\square$

To present our characterization of the operator $\Upsilon(T)$, we will need two more notions.

[3] The notions of a generating default and a reduct used in this paper are different from the standard ones [Reiter, 1980; Gelfond and Lifschitz, 1988]. They are tailored specifically to the needs of constrained default logic.

Definition 3.2 We define the *reduct* of a set of defaults D with respect to a theory T as

$$D_T = \left\{ \frac{\alpha}{\gamma} : \frac{\alpha : M\beta}{\gamma} \in D, T \cup \{\beta,\gamma\} \not\vdash \bot \right\}.$$

We define the operator C by letting

$$C(D,S,T) = \left\{ \beta \wedge \gamma : \frac{\alpha : M\beta}{\gamma} \in GD_D^{(S,T)} \right\}.$$

$\square$

These concepts allow us to give a constructive description of the operator $\Upsilon(T)$.

Theorem 3.3 *Let (D,W) be a default theory and let T be a propositional theory. Then*

$$\Upsilon(T) = (Cn^{D_T}(W), Cn(W \cup C(D, Cn^{D_T}(W), T))).$$

Proof. According to Definition 2.3, $\Upsilon(T)$ is the pair of smallest sets of formulas (S',T') satisfying **CE1–CE3**. It follows from **CE3** that S' is closed under inference rules from D_T. Thus, by the definition of the operator $Cn^A(\cdot)$, $S' = Cn^{D_T}(W)$ satisfies **CE1–CE3**. We need to prove that $T' = Cn(W \cup C(D, Cn^{D_T}(W), T))$ together with $S' = Cn^{D_T}(W)$ satisfies **CE1–CE3**.

We will first prove that $Cn^{D_T}(W) \subseteq Cn(W \cup C(D, Cn^{D_T}(W), T))$. Let $\varphi \in Cn^{D_T}(W)$. Then φ has a proof from W in $PC + D_T$, that is, there is a finite sequence $\varphi_1, \ldots, \varphi_n = \varphi$ such that for every i, $1 \le i \le n$, at least one of the following conditions holds:

1. $\varphi_i \in W$ or φ_i is a substitution instance of an axiom of propositional logic.

2. For some $j,k < i$, φ_i follows from φ_j and φ_k by modus ponens.

3. For some $j < i$, the rule $\frac{\varphi_j}{\varphi_i}$ belongs to D_T.

We will prove by induction on the length of this proof that $\varphi \in Cn(W \cup C(D, Cn^{D_T}(W), T))$. If $\varphi \in W$ or φ is a substitution instance of an axiom of propositional logic then it is clear that $\varphi \in Cn(W \cup C(D, Cn^{D_T}(W), T))$. If φ follows from φ_j and φ_k by modus ponens then, by the inductive hypothesis, $\varphi_j, \varphi_k \in Cn(W \cup C(D, Cn^{D_T}(W), T))$. It follows that $\varphi \in Cn(W \cup C(D, Cn^{D_T}(W), T))$. If φ was obtained by applying an inference rule $\frac{\psi}{\varphi}$ from D_T then $\psi \in Cn^{D_T}(W)$. Hence, there is a default $\frac{\psi : M\beta}{\varphi} \in D$ such that $\psi \in Cn^{D_T}(W)$ and $T \cup \{\beta,\varphi\} \not\vdash \bot$. It follows that $\beta \wedge \varphi \in C(D, Cn^{D_T}(W), T)$ and, consequently, $\varphi \in Cn(W \cup C(D, Cn^{D_T}(W), T))$. Thus, $Cn^{D_T}(W) \subseteq Cn(W \cup C(D, Cn^{D_T}(W), T))$ and **CE1** holds.

The theory $Cn(W \cup C(D, Cn^{D_T}(W), T))$ is closed under propositional provability, so **CE2** holds. To prove **CE3**, let us consider a default $\frac{\alpha : M\beta}{\gamma}$ such that $\alpha \in S' = Cn^{D_T}(W)$ and $T \cup \{\beta,\gamma\} \not\vdash \bot$. It follows from the definition of $C(D,S,T)$ that $\beta \wedge \gamma \in C(D, Cn^{D_T}(W), T) \subseteq$

$Cn(W \cup C(D, Cn^{D_T}(W), T))$. Hence, the pair $(Cn^{D_T}(W), Cn(W \cup C(D, Cn^{D_T}(W), T)))$ satisfies **CE1**–**CE3**.

Let us assume now that a pair of theories (S', T') satisfies **CE1**–**CE3**. Then it is easy to see that $Cn^{D_T}(W) \subseteq S'$. By **CE1**, $W \subseteq T'$. Let us consider a formula $\beta \wedge \gamma \in C(D, Cn^{D_T}(W), T)$. Then there is a default $\frac{\alpha : M\beta}{\gamma} \in D$ such that $\alpha \in Cn^{D_T}(W) \subseteq S'$ and $T \cup \{\beta, \gamma\} \not\vdash \bot$. It follows from **CE3** that $\beta \wedge \gamma \in T'$. Hence, $C(D, Cn^{D_T}(W), T) \subseteq T'$. Since, by **CE2**, $T' = Cn(T')$, then $Cn(W \cup C(D, Cn^{D_T}(W), T)) \subseteq T'$ and we are done. □

As a corollary, we get a characterization of constrained extensions.

Corollary 3.4 (S, T) *is a constrained extension of a default theory* (D, W) *if and only if*

$$T = Cn(W \cup C(D, Cn^{D_T}(W), T)) \quad (2)$$

and $S = Cn^{D_T}(W)$. □

This corollary makes it explicit that constrained default logic works in two stages. In the first stage, possible sets of assumptions (constraints, as referred to in [Schaub, 1992]) are established as solutions of the fixpoint equation (2). Then, each of them uniquely determines the corresponding proper constrained extention.

The fixpoint equation (2) implies that all assumptions (theory T) needed to support a proper constrained extension S in constrained default logic must be "reprovable". This might be regarded as a weakness of constrained default logic. In all versions of default logic it is required that formulas in the extensions have justifications (in terms of proofs from W by means of applicable defaults). But it is arguable whether the same should be required of assumptions "β is possible" which make defaults applicable.

Corollary 3.4 allows us to strengthen a result by Schaub on pairwise maximality of constrained extensions of a default theory.

Theorem 3.5 *If* (S, T) *and* (S', T') *are constrained extensions of a default theory* (D, W) *and* $T \subseteq T'$, *then* $T = T'$ *and* $S = S'$ *(in particular, "T-parts" of constrained extensions form an antichain).*

Proof. First, observe that if (S', T') is a constrained extension of (D, W), then T' is inconsistent if and only if W is inconsistent. Hence, if T' is inconsistent then T is inconsistent as well and, consequently, $T = T'$ and $S = S'$. Assume then that T' is consistent.

Since (S, T) is a constrained extension of (D, W), by Corollary 3.2 we have $S = Cn^{Mon(GD_D^{(S,T)})}(W)$. Moreover, for every default $\frac{\alpha : M\beta}{\gamma} \in GD_D^{(S,T)}$, $\beta \wedge \gamma \in T$ and, since $T \subseteq T'$, $\beta \wedge \gamma \in T'$. Since T' is consistent, it follows that $Mon(GD_D^{(S,T)}) \subseteq D_{T'}$. Hence,

$$S = Cn^{Mon(GD_D^{(S,T)})}(W) \subseteq Cn^{D_{T'}}(W) = S'.$$

Schaub [1992] proved that if (S, T) and (S', T') are constrained extensions of a default theory and $S \subseteq S'$ and $T \subseteq T'$, then $S = S'$ and $T = T'$. □

Theorem 3.5 implies bounds on the number of constrained extensions of a default theory.

Corollary 3.6 *Let* D *be a set consisting of* n *defaults. Then for every* $W \subseteq \mathcal{L}$, *the default theory* (D, W) *has at most*

$$\binom{n}{\lceil n/2 \rceil}$$

constrained extensions.

Proof. Theorem 3.5 implies that the family $\mathcal{T} = \{T : (S, T) \text{ is a constrained extension of } (D, W)\}$ is an antichain. By Theorem 3.1, each set $T \in \mathcal{T}$ is determined by the set $GD_D^{(S,T)}$. Since $\mathcal{T}$ is an antichain, the family $\{GD_D^{(S,T)} : T \in \mathcal{T}\}$ is also an antichain. It is well known that the size of the largest antichain in the algebra of subsets of an n-element set has at most $\binom{n}{\lceil n/2 \rceil}$ elements. Hence, the assertion follows. □

The second characterization of constrained extensions that we present in this section is closely related to the property of semi-monotonicity of constrained default logic. It exploits the fact that constrained extensions can be produced by processing defaults according to any well-ordering. This is very similar to the corresponding property of normal default theories in the logic of Reiter (Theorem 4.3, [Marek and Truszczyński, 1993]). In fact, our characterization of constrained default logic provides an alternative argument that for normal default theories constrained and standard default logics coincide.

We assume that the set of the atoms of our language $\mathcal{L}$ is denumerable. Consequently, the set of all defaults over the language $\mathcal{L}$ is denumerable.

Let (D, W) be a default theory and $\prec$ a well-ordering of D. We define an ordinal $\eta_\prec$. For every ordinal $\xi < \eta_\prec$ we define a set of defaults AD_ξ and a default d_ξ. We also define a set of defaults $AD_\prec$. We proceed as follows:
If the sets AD_ξ, $\xi < \alpha$, have been defined but $\eta_\prec$ has not been defined then

1. If there is no default $d \in D \setminus \bigcup_{\xi < \alpha} AD_\xi$ such that:
   - **(a)** $W \cup c(\bigcup_{\xi<\alpha} AD_\xi) \cup j(\bigcup_{\xi<\alpha} AD_\xi) \cup j(d) \cup \{c(d)\}$ is consistent, and
   - **(b)** $W \cup c(\bigcup_{\xi<\alpha} AD_\xi) \vdash p(d)$,

   then $\eta_\prec = \alpha$.

2. Otherwise, define d_α to be the $\prec$-least default $d \in D \setminus \bigcup_{\xi<\alpha} AD_\xi$ such that the conditions **(a)** and **(b)** above hold. Then set $AD_\alpha = \bigcup_{\xi<\alpha} AD_\xi \cup \{d_\alpha\}$.

When the construction terminates, put $AD_\prec = \bigcup_{\xi<\eta_\prec} AD_\xi$, $S_\prec = Cn(W \cup c(AD_\prec))$ and $T_\prec = Cn(W \cup c(AD_\prec) \cup j(AD_\prec))$.

This construction has the following property.

Theorem 3.7 *Let (D,W) be a default theory and $\prec$ a well-ordering of D. Then $(S_\prec,T_\prec)$ is a constrained extension of (D,W) and $AD_\prec = GD_D^{(S_\prec,T_\prec)}$.*

Proof. It is easy to see that if W is inconsistent then $AD_\prec = \emptyset$ and $S_\prec = T_\prec = Cn(W) = \mathcal{L}$. Thus, the assertion follows. Assume now that W is consistent. Then for every $\xi < \eta_\prec$, $W \cup c(AD_\xi) \cup j(AD_\xi)$ is consistent. Consequently, $W \cup c(AD_\prec) \cup j(AD_\prec)$ is consistent. Thus, $T_\prec$ is consistent.

Let us consider a default $d = \frac{\alpha:M\beta}{\gamma} \in AD_\prec$. We have $\beta \in j(AD_\prec)$ and $\gamma \in c(AD_\prec)$. Thus, $\{\beta,\gamma\} \subseteq T_\prec$ and, since $T_\prec$ is consistent, $T_\prec \cup \{\beta,\gamma\}$ is consistent. Moreover, since $d \in AD_\prec$, for some $\xi < \eta_\prec$, $W \cup c(\bigcup_{\lambda<\xi} AD_\lambda) \vdash \alpha$. Consequently, $\alpha \in S_\prec$. Hence, $d \in GD_D^{(S_\prec,T_\prec)}$.

Consider now a default $d = \frac{\alpha:M\beta}{\gamma} \in D \setminus AD_\prec$. Then one of the conditions **(a)**, **(b)** above fails for d, that is, either $T_\prec \cup \{\beta,\gamma\}$ is inconsistent or $S_\prec \not\vdash \alpha$. Hence, $d \notin GD_D^{(S_\prec,T_\prec)}$. Thus, $AD_\prec = GD_D^{(S_\prec,T_\prec)}$.

The last equality implies that

$$\begin{aligned} T_\prec &= Cn(W \cup c(GD_D^{(S_\prec,T_\prec)}) \cup j(GD_D^{(S_\prec,T_\prec)})) \\ &= Cn(W \cup \{\beta \wedge \gamma : \frac{\alpha: M\beta}{\gamma} \in GD_D^{(S_\prec,T_\prec)}\}) \\ &= Cn(W \cup C(D,S_\prec,T_\prec)). \end{aligned}$$

Thus, to end the proof we need to show that $S_\prec = Cn^{D_{T_\prec}}(W)$. Let us notice that the condition **(b)** above implies that $Cn^{Mon(AD_\prec)}(W) = Cn(W \cup c(AD_\prec)) = S_\prec$. Thus, $S_\prec = Cn^{Mon(GD_D^{(S_\prec,T_\prec)})}(W)$. Let $A = \{\frac{\alpha:M\beta}{\gamma} \in D : T_\prec \cup \{\beta,\gamma\} \not\vdash \bot\}$. Then $D_{T_\prec} = Mon(A)$ and $GD_D^{(S_\prec,T_\prec)} = \{d \in A : p(d) \in S_\prec\}$. Hence, $GD_D^{(S_\prec,T_\prec)} \subseteq A$ and

$$S_\prec = Cn^{Mon(GD_D^{(S_\prec,T_\prec)})}(W)$$

$$\subseteq Cn^{Mon(A)}(W) = Cn^{D_{T_\prec}}(W).$$

Thus, to show that $S_\prec = Cn^{D_{T_\prec}}(W)$, we need to prove that $Cn^{D_{T_\prec}}(W) \subseteq Cn^{Mon(GD_D^{(S_\prec,T_\prec)})}(W)$. To this end, let us consider a formula $\varphi \in Cn^{D_{T_\prec}}(W)$. The formula φ has a proof from W in $PC + D_{T_\prec}$. One can prove by induction on the length of this proof that $\varphi \in Cn^{Mon(GD_D^{(S_\prec,T_\prec)})}(W)$ (due to space restrictions, we omit the details of the argument). □

The converse result is also true.

Theorem 3.8 *Let (S,T) be a constrained extension of a default theory (D,W). Then for any well-ordering $\prec$ of D such that defaults from $GD_D^{(S,T)}$ precede all other defaults, $AD_\prec = GD_D^{(S,T)}$, $S_\prec = S$ and $T_\prec = T$.* □

This result will be proved in a full version of the paper.

4 Computational aspects of constrained default logic

Corollary 3.4 allows us to design an algorithm for computing all constrained extensions for a finite default theory (D,W). Let us consider a set of defaults $A \subseteq D$. Let $S = Cn^{Mon(A)}(W)$, $B = \{d \in A : p(d) \in S\}$, $C = \{\beta \wedge \gamma : \frac{\alpha:M\beta}{\gamma} \in B\}$, $T = Cn(W \cup C)$ and $E = \{\frac{\alpha:M\beta}{\gamma} \in D : T \cup \{\beta,\gamma\} \not\vdash \bot\}$. It is easy to see that if $E = A$ then (S,T) is a constrained extension for (D,W). Indeed, $D_T = Mon(E)$. Thus, if $E = A$ then $Mon(A) = D_T$ and $S = Cn^{D_T}(W)$. Moreover, $B = GD_D^{(S,T)}$. Hence, $C = C(D,S,T) = C(D, Cn^{D_T}(W), T)$ and $T = Cn(W \cup C) = Cn(W \cup C(D, Cn^{D_T}(W), T))$. Thus, (S,T) is a constrained extension for (D,W). Considering all sets $A \subseteq D$, we will get all constrained extensions of (D,W). Indeed, if (S,T) is a constrained extension then it will be found by considering $A = \{\frac{\alpha:M\beta}{\gamma} \in D : T \cup \{\beta,\gamma\} \not\vdash \bot\}$. Thus, we have the following algorithm:

For every $A \subseteq D$

1. compute U such that $Cn(U) = Cn^{Mon(A)}(W)$ (the theory $Cn^{Mon(A)}(W)$ is infinite, however U is finite — see [Marek and Truszczyński, 1993]),
2. compute $B = \{d \in A : U \vdash p(d)\}$, and let $C = \{\beta \wedge \gamma : \frac{\alpha:M\beta}{\gamma} \in B\}$,
3. compute $E = \{\frac{\alpha:M\beta}{\gamma} \in D : W \cup C \cup \{\beta,\gamma\} \not\vdash \bot\}$,
4. if $E = A$ then output $(Cn(U), Cn(W \cup C))$ as a constrained extension of (D,W).

The complexity of the above algorithm is determined by the number of calls to a propositional consistency checking procedure. Assume that the number of defaults in D is n. Given $A \subseteq D$, we need at most n^2 calls to such a procedure to compute U, at most n calls to compute B and n calls to compute E. Hence, for every $A \subseteq D$, we need $O(n^2)$ calls to a propositional consistency checking procedure, and $O(n^2 2^n)$ calls to such a procedure for the whole algorithm.

We have the following complexity result (see [Garey and Johnson, 1979] for a discussion of complexity classes).

Theorem 4.1 *The following problems:*

IN-SOME *Given a finite default theory (D,W) and a formula φ, decide if φ is in some constrained extension for (D,W),*

NOT-IN-ALL *Given a finite default theory (D,W) and a formula φ, decide if there is a constrained extension for (D,W) not containing φ,*

are Σ_2^P-complete. The problem

IN-ALL *Given a finite default theory* (D, W) *and a formula* φ*, decide if* φ *is in all constrained extensions of* (D, W)*,*

is Π_2^P*-complete.*

Proof. To verify that a formula φ belongs to some (does not belong to all, resp.) constrained extensions of (D, W), we can nondeterministically guess a set of defaults $A \subseteq D$, verify that $(Cn(U), Cn(W \cup C))$ is a constrained extension for (D, W) (U and C are as defined in the above algorithm) and verify that $U \vdash \varphi$ ($U \not\vdash \varphi$, resp.), what requires one more call to a propositional consistency checking procedure. It follows that the problems IN-SOME and NOT-IN-ALL are in Σ_2^P. Since the problem NOT-IN-ALL is in Σ_2^P, the problem IN-ALL is in Π_2^P. Observe next that S is an extension for a normal default theory if and only if (S, S) is a constrained extension for this theory (see [Schaub, 1992]). Hence, the hardness of all three problems in their respective complexity classes follows from the fact that the problems IN-SOME, NOT-IN-ALL and IN-ALL for extensions of normal default theories are Σ_2^P-hard and Π_2^P-hard, respectively (see [Gottlob, 1992]). □

5 Connections between constrained and rational default logics

First, we will show that rational extensions determine constrained extensions. That is, we will show that for every rational extension S of a default theory, there is T such that (S, T) is a constrained extension of this theory.

Theorem 5.1 *Let* E *be a rational extension for a default theory* (D, W) *and let* $A \in \mathcal{MA}(D, W, E)$ *be such that* $E = Cn^{Mon(A)}(W)$ $(= Cn(W \cup c(A)))$*. Let* $C = Cn(E \cup j(A))$ $(= Cn(W \cup c(A) \cup j(A)))$*. Then* (E, C) *is a constrained extension of* (D, W)*.*

Proof. We need to show that $\Upsilon(C) = (E, C)$. We will prove first that pair (E, C) satisfies **CE1**–**CE3** with respect to C. The conditions **CE1** and **CE2** are obviously satisfied. Now, let a default $d = \frac{\alpha : M\beta}{\gamma}$ be such that $\alpha \in E$ and $C \cup \{\beta, \gamma\}$ is consistent. Since $C \cup \{\beta, \gamma\}$ is consistent, then $E \cup j(A \cup \{d\}) = E \cup j(A) \cup \{\beta\}$ is consistent. Moreover, since $p(A) \subseteq Cn^{Mon(A)}(W) = E$ and $\alpha \in E$, we have $p(A \cup \{d\}) = p(A) \cup \{\alpha\} \subseteq E = Cn^{Mon(A)}(W) \subseteq Cn^{Mon(A \cup \{d\})}(W)$. Thus, $(A \cup \{d\}) \in \mathcal{A}(D, W, E)$. By the maximality of A, $d \in A$. Hence, $\gamma \in c(A) \subseteq E$ and $\{\beta, \gamma\} \subseteq c(A) \cup j(A) \subseteq C$. It follows that $\gamma \in E$ and $\beta \wedge \gamma \in Cn(C) = C$ and **CE3** holds.

Now, we need to prove that if a pair (S, T) satisfies **CE1**–**CE3** with respect to C then $E \subseteq S$ and $C \subseteq T$. Let us notice that, since $c(A) \cup j(A) \subseteq C$, for $d = \frac{\alpha : M\beta}{\gamma} \in A$, $C \cup \{\beta, \gamma\} = C$. By **AS1**, $j(A) = \emptyset$ or $j(A) \cup E$ is consistent. Thus, we have either $C = Cn(E \cup j(A))$ is consistent or $A = \emptyset$ (let us recall that we do not consider justification-free defaults). In both cases for $d = \frac{\alpha : M\beta}{\gamma} \in A$, $C \cup \{\beta, \gamma\}$ is consistent. Thus, **CE3** implies that S is closed under inference rules from $Mon(A)$. Moreover, by **CE1**, $W \subseteq S$ and, by **CE2**, S is closed under propositional provability. It follows that $E = Cn^{Mon(A)}(W) \subseteq S$. We have $p(A) \subseteq E \subseteq S$. So, it follows from **CE3** and **CE2** that $j(A) \cup c(A) \subseteq T$. Since, by **CE1**, $W \subseteq T$ and, by **CE2**, $T = Cn(T)$, then $C \subseteq T$ and we are done. □

The converse statement is not true. Every default theory has at least one constrained extension [Schaub, 1992] and there are default theories that do not have rational extensions [Mikitiuk and Truszczyński, 1993]. We will show that for seminormal default theories an exact one-to-one correspondence between rational and constrained extensions can be established. But first we will prove an auxiliary result.

Theorem 5.2 *Let* (S, T) *be a constrained extension of a default theory* (D, W)*. Then* $GD_D^{(S,T)} \in \mathcal{A}(D, W, S)$*.*

Proof. It follows from Theorem 3.1 that $T = Cn(S \cup j(GD_D^{(S,T)}))$. If T is consistent then $j(GD_D^{(S,T)}) \cup S$ is also consistent. If T is inconsistent then it follows from the definition of $GD_D^{(S,T)}$ that $GD_D^{(S,T)} = \emptyset$, so $j(GD_D^{(S,T)}) = \emptyset$. Thus, in both cases **AS1** holds.

According to the definition of $GD_D^{(S,T)}$, $p(GD_D^{(S,T)}) \subseteq S$. By Corollary 3.2, $S = Cn^{Mon(GD_D^{(S,T)})}(W)$. Hence, **AS2** also holds. □

Theorem 5.3 *Let* (S, T) *be a constrained extension of a seminormal default theory* (D, W)*. Then* S *is a rational extension of* (D, W) *and* $GD_D^{(S,T)} \in \mathcal{MA}(D, W, S)$*.*

Proof. By Theorem 5.2, $GD_D^{(S,T)} \in \mathcal{A}(D, W, S)$ and, by Corollary 3.2, $S = Cn^{Mon(GD_D^{(S,T)})}(W)$. Thus, we need to prove the maximality of $GD_D^{(S,T)}$ in $\mathcal{A}(D, W, S)$ only. Since every default in D is seminormal, we can rewrite the definitions of $GD_D^{(S,T)}$, D_T and $C(D, S, T)$ as

$$GD_D^{(S,T)} = \left\{ \frac{\alpha : M\beta}{\gamma} \in D : \alpha \in S, T \cup \{\beta\} \not\vdash \bot \right\},$$

$$D_T = \left\{ \frac{\alpha}{\gamma} : \frac{\alpha : M\beta}{\gamma} \in D, T \cup \{\beta\} \not\vdash \bot \right\},$$

$$C(D, S, T) = \left\{ \beta : \frac{\alpha : M\beta}{\gamma} \in GD_D^{(S,T)} \right\} = j(GD_D^{(S,T)}).$$

Let us denote $GD_D^{(S,T)}$ by A. Since, by Corollary 3.4, $S = Cn^{D_T}(W)$ and $T = Cn(W \cup C(D, Cn^{D_T}(W), T))$, we get $T = Cn(W \cup j(A))$.

Let us consider now a default $d = \frac{\alpha : M\beta}{\gamma} \in D \setminus A$. We have either $\alpha \notin S = Cn^{Mon(A)}(W)$ or $T \cup \{\beta\}$ is inconsistent. If $T \cup \{\beta\}$ is inconsistent then $W \cup j(A) \cup \{\beta\} = W \cup j(A \cup \{d\})$ is inconsistent and the set $A \cup \{d\}$ does not satisfy **AS1** (since d is seminormal, $j(\{d\}) \neq \emptyset$). If $\alpha \notin Cn^{Mon(A)}(W)$ then $Cn^{Mon(A \cup \{d\})}(W) = Cn^{Mon(A)}(W)$ and $p(A \cup \{d\}) = p(A) \cup \{\alpha\} \not\subseteq Cn^{Mon(A \cup \{d\})}(W)$, so $A \cup \{d\}$ does not satisfy **AS2**. By Proposition 4.2 from an extended version of [Mikitiuk and Truszczyński, 1993], $A \in \mathcal{MA}(D, W, S)$ and we are done. □

Schaub ([1992]) proved that every default theory has a constrained extension, so we have the following corollary (proved first by other methods in an extended version of [Mikitiuk and Truszczyński, 1993]).

Corollary 5.4 *Every seminormal default theory has a rational extension.* □

6 The case of justification-free defaults

We close this paper with a remark that constrained default logic can be extended to cover the case of default theories that may contain justification-free defaults. To this end, one has to replace **CE3** by the following two conditions:

CE3$_1$ For any default $\frac{\alpha : M\beta}{\gamma} \in D$, if $\alpha \in S'$ and $T \cup \{\beta, \gamma\}$ is consistent then $\gamma \in S'$ and $\beta \wedge \gamma \in T'$.

CE3$_2$ For any default $\frac{\alpha :}{\gamma} \in D$, if $\alpha \in S'$ and $T \cup \{\gamma\}$ is consistent then $\gamma \in S'$.

The definitions of $GD_D^{(S,T)}$, D_T and $C(D, S, T)$ must be modified in the same way. Under such modifications all results presented in this paper, except for Theorem 5.1, remain true. Moreover, the following, slightly modified version of Theorem 5.1 holds (the assumption of consistency of a rational extension is added).

Theorem 6.1 *Let E be a consistent rational extension for a default theory (D, W) and let $A \in \mathcal{MA}(D, W, E)$ be such that $E = Cn^{Mon(A)}(W)$ $(= Cn(W \cup c(A)))$. Let $C = Cn(E \cup j(A))$ $(= Cn(W \cup c(A) \cup j(A)))$. Then (E, C) is a constrained extension of (D, W).*

Proofs of these results will be included in a full version of the paper.

7 Conclusions

In this paper we showed that constrained and rational default logics are closely related. While Reiter's default logic and constrained default logic coincide on the class of normal default theories, rational and constrained default logics coincide on a much wider class of seminormal default theories.

We showed that basic problems of reasoning with constrained extensions are complete for the second level of the polynomial hierarchy (with the exception of the existence of an extension problem, which is trivially in P). We also proposed algorithms to compute constrained extensions.

Constrained default logic was originally introduced only for default theories without justification-free defaults. In the paper, we proposed a modification of the original definition of Schaub, which allows for defaults to be justification-free. Under our definition, all major properties of constrained default logic remain true.

References

[Besnard, 1989] P. Besnard. *An introduction to default logic.* Springer-Verlag, Berlin, 1989.

[Brewka, 1991a] G. Brewka. Cumulative default logic: in defense of nonmonotonic inference rules. *Artificial Intelligence*, 50:183–205, 1991.

[Brewka, 1991b] G. Brewka. *Nonmonotonic reasoning: logical foundations of commonsense.* Cambridge University Press, Cambridge, UK, 1991.

[Garey and Johnson, 1979] M.R. Garey and D.S. Johnson. *Computers and intractability; a guide to the theory of NP-completeness.* W.H. Freeman, 1979.

[Gelfond and Lifschitz, 1988] M. Gelfond and V. Lifschitz. The stable semantics for logic programs. In R. Kowalski and K. Bowen, editors, *Proceedings of the 5th international symposium on logic programming*, pages 1070–1080, Cambridge, MA., 1988. MIT Press.

[Gelfond *et al.*, 1991] M. Gelfond, V. Lifschitz, H. Przymusinska, and M. Truszczyński. Disjunctive defaults. In *Second international conference on principles of knowledge representation and reasoning, KR '91*, Cambridge, MA, 1991.

[Gottlob and Mingyi, 1994] G. Gottlob and Z. Mingyi. Cumulative default logic: finite characterization, algorithms, and complexity. *Artificial Intelligence*, 69:329–345, 1994.

[Gottlob, 1992] G. Gottlob. Complexity results for nonmonotonic logics. *Journal of Logic and Computation*, 2:397–425, 1992.

[Marek and Truszczyński, 1993] W. Marek and M. Truszczyński. *Nonmonotonic logics; context-dependent reasoning.* Berlin: Springer-Verlag, 1993.

[Mikitiuk and Truszczyński, 1993] A. Mikitiuk and M. Truszczyński. Rational default logic and disjunctive logic programming. In A. Nerode and L. Pereira, editors, *Logic programming and non-monotonic reasoning*, pages 283–299. MIT Press, 1993.

[Poole, 1989] D. Poole. What the lottery paradox tells us about default reasoning. In *Proceedings of the 2nd conference on principles of knowledge representation and reasoning, KR '89*, pages 333–340, San Mateo, CA., 1989. Morgan Kaufmann.

[Reiter, 1980] R. Reiter. A logic for default reasoning. *Artificial Intelligence*, 13:81–132, 1980.

[Schaub, 1992] T. Schaub. *Considerations on Default Logics.* PhD thesis, Technischen Hochschule Darmstadt, 1992.