Computing minimal models, stable models and answer sets

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Abstract

We propose and study algorithms to compute minimal models, stable models and answer sets of t-CNF theories, and normal and disjunctive t-programs. We are especially interested in algorithms with *non-trivial* worst-case performance bounds. The bulk of the paper is concerned with the classes of 2- and 3-CNF theories, and normal and disjunctive 2- and 3-programs, for which we obtain significantly stronger results than those implied by our general considerations. We show that one can find all minimal models of 2-CNF theories and all answer sets of disjunctive 2-programs in time $O(m1.4422..^n)$. Our main results concern computing stable models of normal 3-programs, minimal models of 3-CNF theories and answer sets of disjunctive 3-programs. We design algorithms that run in time $O(m1.6701..^n)$, in the case of the first problem, and in time $O(mn^2 2.2782..^n)$, in the case of the latter two. All these bounds improve by exponential factors the best algorithms known previously. We also obtain closely related upper bounds on the number of minimal models, stable models and answer sets a t-CNF theory, a normal t-program or a disjunctive t-program may have.

KEYWORDS: Stable models, answer sets, minimal models, disjunctive logic programs

1 Introduction

We study the problem of computing *minimal* models of CNF theories, *stable* models of normal logic programs and *answer sets* of disjunctive logic programs. We are especially interested in algorithms, for which we can derive *non-trivial* worst-case performance bounds. Our work builds on studies of algorithms to compute models of propositional CNF theories (Kullmann 1999) and improves on our earlier study of algorithms to compute stable models (Lonc and Truszczyński 2003).

In the paper, by $At(T)$ and $At(P)$ we denote the set of atoms occurring in a theory T or a program P, respectively. We represent models of propositional theories, stable models of normal logic programs and answer sets of disjunctive

logic programs as sets of atoms that are true in these models and answer sets. When discussing the complexity of algorithms, we consistently write n for the number of atoms and m for the size (the total number of atom occurrences) of an input theory T or an input program P, even when we do not explicitly mention them.

Propositional logic with the minimal-model semantics (propositional circumscription) (McCarthy 1980; Lifschitz 1988), logic programming with stable-model semantics (Gelfond and Lifschitz 1988) and disjunctive logic programming with the answer-set semantics (Gelfond and Lifschitz 1991) are among the most commonly studied and broadly used knowledge representation formalisms (we refer the reader to (Marek and Truszczyński 1993; Brewka et al. 1997) for a detailed treatment of these logics and additional pointers to literature). Recently, they have been receiving much attention due to their role in *answer-set programming* (ASP) — an emerging declarative programming paradigm. Fast algorithms to compute minimal models, stable models and answer sets are essential for the computational effectiveness of propositional circumscription, logic programming and disjunctive logic programming as answer-set programming systems.

These computational tasks can be solved by a "brute-force" straightforward search. We can compute all minimal models of a CNF theory T by checking for each set $M \subseteq At(T)$ whether M is a minimal model of T. To this end, we test first whether M is a model of T. If not, M is not a minimal model of T and we move on to the next subset of $At(T)$. Otherwise, we test whether any *proper* subset of M is a model of T. If the answer is yes, then M is not a minimal model of T and we consider another subset of $At(T)$. If the answer is no, then M is a minimal model of T. Testing whether a subset of $At(T)$ is a model of T can be accomplished in $O(m)$ steps. Thus, if $|M| = i$, we can verify whether M is a minimal model of T in time $O(m2^i)$. Checking all sets of cardinality i requires $O(\binom{n}{i}m2^i)$ steps and checking all sets — $O(\sum_{i=0}^{n}\binom{n}{i}m2^i) = O(m3^n)$. Thus, this brute-force approach to compute minimal models works in time $O(m3^n)$.

To determine whether a set of atoms M is an answer set of a disjunctive logic program P we need to verify whether M is a minimal model of the *reduct of P with respect to M* (Gelfond and Lifschitz 1991) or, equivalently, whether M is a minimal model of the propositional theory that corresponds to the reduct. Thus, a similar argument as before demonstrates that answer sets of a finite propositional disjunctive logic program P can also be computed in time $O(m3^n)$.

In the case of stable models we can do better. The task of verifying whether a set of atoms M is a stable model of a finite propositional logic program P can be accomplished in time $O(m)$. Consequently, one can compute all stable models of P in $O(m2^n)$ steps using an exhaustive search through all subsets of $At(P)$, and checking for each of them whether it is a stable model of P.

A fundamental question, and the main topic of interest in this paper, is whether there are algorithms for the three computational problems discussed here with better worst-case performance bounds.

We note that researchers proposed several algorithms to compute minimal models of propositional CNF theories (Ben-Eliyahu and Palopoli 1994; Niemelä 1996), stable models of logic programs (Simons et al. 2002) and answer sets of disjunctive logic

programs (Eiter et al. 2000). Some implementations based on these algorithms, for instance, *smodels* (Simons et al. 2002) and *dlv* (Eiter et al. 2000), perform very well in practice. However, so far very little is known about the worst-case performance of these implementations.

In this paper, we study the three computational problems discussed earlier. We focus our considerations on *t-CNF theories* and *t-programs*, that is, theories and programs, respectively, consisting of clauses containing no more than t literals. Such theories and programs arise often in the context of search problems. Given a search problem, we often encode its specifications as a (disjunctive) DATALOG$^\neg$ program (or, a set of *propositional schemata* — universally quantified clauses in some function-free language (East and Truszczyński 2006)). Propositional program (CNF theory) corresponding to a *concrete* instance of the search problem is then obtained by grounding the (disjunctive) DATALOG$^\neg$ rules (propositional schemata) with constants appearing in the descriptions of the instance. Since the initial program (set of propositional schemata) is independent of particular problem instances, grounding results in propositional programs (CNF theories) with clauses of bounded length, in other words, in t-programs (t-theories) for some, typically quite small, value of t. In fact in many cases, $t = 2$ or $t = 3$ (it is so, for instance, for problems discussed in (Marek and Truszczyński 1999; Niemelä 1999), once the so called domain predicates are simplified away).

In our earlier work, we studied the problem of computing stable models of normal t-programs (Lonc and Truszczyński 2003). We obtained some general results in the case of an arbitrary $t \geq 2$ and were able to strengthen them for two special cases of $t = 2$ and $t = 3$. We presented an algorithm to compute all stable models of a normal 2-program in time $O(m3^{n/3}) = O(m1.4422..^n)$ and showed its asymptotic optimality. We proposed a similar algorithm for the class of normal 3-programs and proved that its running time is $O(m1.7071..^n)$. Finally, we applied the techniques developed in this paper to obtain a non-trivial worst-case performance bound for *smodels*, when input programs are restricted to be 2-programs.

In this paper, we improve on our results from (Lonc and Truszczyński 2003) and extend them to the problems of computing minimal models of t-CNF theories and answer sets of disjunctive t-programs. We present results concerning the case of an arbitrary $t \geq 2$ but the bulk of the paper is devoted to 2- and 3-CNF theories and 2- and 3-programs, for which we significantly strengthen our general results.

First, we show how to find all minimal models of 2-CNF theories and all answer sets of disjunctive 2-programs in time $O(m1.4422..^n)$, generalizing a similar result we obtained earlier for computing stable models of normal 2-programs. Our main results concern computing stable models of normal 3-programs, minimal models of 3-CNF theories and answer sets of disjunctive 3-programs. We design algorithms that run in time $O(m1.6701..^n)$, for the first problem, and in time $O(mn^2 2.2782..^n)$, for the latter two. These bounds improve by exponential factors the best algorithms known previously. We also obtain closely related upper bounds on the number of minimal models, stable models and answer sets a 2- or 3-theory or program may have.

Our paper is organized as follows. In the next section we state the main results

of the paper. In the remainder of the paper we prove them. First, in Section 3, we present an auxiliary algorithm min^+ that, given an arbitrary CNF theory T, outputs a family of sets containing all minimal models of T. We also derive some general bounds on the performance of the algorithm min^+ and on the number of sets it outputs. In the following section, we adapt the algorithm min^+ to each of the computational tasks of interest to us: finding all minimal models of CNF theories, stable models of normal programs and answer sets of disjunctive programs. In Sections 5 and 6, we specialize the algorithms from Section 4 to the case of t-CNF theories (normal t-programs and disjunctive t-programs) for $t = 2$ and 3, respectively. In Section 7, we outline the proof of a main technical lemma, on which all results concerning 3-CNF theories, 3-programs and disjunctive 3-programs depend. We present a detailed proof of this result in the appendix. In Section 8, we discuss the case of an arbitrary $t \geq 2$. In Section 9, we discuss lower bounds on the numbers of minimal models (stable models, answer sets) of t-CNF theories (normal t-programs, disjunctive t-programs). We use these bounds in the last section of the paper to discuss the optimality of our results and to identify some open problems for future research.

2 Main results

We will now present and discuss the main results of our paper. We start by stating two theorems that deal with minimal models of 2-CNF theories, stable models of (normal) 2-programs and answer sets of disjunctive 2-programs. The results concerning stable models of 2-programs were first presented in (Lonc and Truszczyński 2003). The results about minimal models of 2-CNF theories and answer sets of disjunctive 2-programs are new.

Theorem 1
There are algorithms to compute all minimal models of 2-CNF theories, stable models of 2-programs and answer sets of disjunctive 2-programs, respectively, that run in time $O(m3^{n/3}) = O(m1.4422..^n)$.

Theorem 2
Every 2-CNF theory (2-program and disjunctive 2-program, respectively) has at most $3^{n/3} = 1.4422..^n$ minimal models (stable models, answer-sets, respectively).

There are 2-CNF theories, 2-programs and disjunctive 2-programs with n atoms and with $\Omega(3^{n/3})$ minimal models, stable models and answer sets, respectively. Thus, the bounds provided by Theorems 1 and 2 are optimal.

Next, we present results concerning 3-CNF theories, 3-programs and disjunctive 3-programs. These results constitute the main contribution of our paper. As in the previous case, we obtain a common upper bound for the number of minimal models, stable models and answer sets of 3-CNF theories, 3-programs and disjunctive 3-programs, respectively. Our results improve the bound on the number of stable models of 3-programs from (Lonc and Truszczyński 2003) and, to the best of our knowledge, provide the first non-trivial bounds on the number of minimal models of 3-CNF theories and answer sets of disjunctive programs.

Theorem 3
Every 3-CNF theory T (every normal 3-program P and every disjunctive 3-program P, respectively) has at most $1.6701..^n$ minimal models (stable models, answer-sets, respectively).

For 2-CNF theories and 2-programs, the common bound on the number of minimal models, stable models and answer sets, appeared also as an exponential factor in formulas estimating the running time of algorithms to compute the corresponding objects. In contrast, we find that there is a difference in how fast we can compute stable models of normal 3-programs as opposed to minimal models of 3-CNF theories and answer sets of disjunctive 3-programs. The reason seems to be that the problem to check whether a set of atoms is a stable model of a normal program is in P, while the problems of deciding whether a set of atoms is a minimal model of a 3-CNF theory or an answer set of a disjunctive 3-program are co-NP complete (Cadoli and Lenzerini 1994; Eiter and Gottlob 1995). For the problem of computing stable models of normal 3-programs we have the following result. It constitutes an exponential improvement on the corresponding result from (Lonc and Truszczyński 2003).

Theorem 4
There is an algorithm to compute all stable models of normal 3-programs that runs in time $O(m1.6701..^n)$.

Our results concerning computing minimal models of 3-CNF theories and answer sets of disjunctive 3-programs are weaker. Nevertheless, in each case they provide an exponential improvement over the trivial bound of $O(m3^n)$ and, to the best of our knowledge, they offer currently the best asymptotic bounds on the performance of algorithms for these two problems.

Theorem 5
There is an algorithm to compute all minimal models of 3-CNF theories and answer sets of disjunctive 3-programs, respectively, that runs in time $O(mn^2 2.2782..^n)$.

Proving Theorems 1 - 5 is our main objective for the remainder of this paper.

3 Technical preliminaries and an auxiliary algorithm

We begin by presenting and analyzing an auxiliary algorithm that, given a CNF theory T computes a superset of the set of all minimal models of T.

In the paper, we consider only CNF theories with no clause containing multiple occurrences of the same literal, and with no clause containing both a literal and its dual. The first assumption allows us to treat clauses interchangingly as disjunctions of their literals or as *sets* of their literals.

Let T be a CNF theory. By $Lit(T)$ we denote the set of all literals built of atoms in $At(T)$. For a literal ω, by $\overline{\omega}$ we denote the literal that is ω's dual. That is, for an atom a we set $\overline{a} = \neg a$ and $\overline{\neg a} = a$.

For a set of literals $L \subseteq Lit(T)$, we define:

$$\overline{L} = \{\overline{\omega} : \omega \in L\}, \quad L^+ = At(T) \cap L \quad \text{and} \quad L^- = At(T) \cap \overline{L}.$$

A set of literals L is *consistent* if $L^+ \cap L^- = \emptyset$. A set of atoms $M \subseteq At(T)$ is *consistent* with a set of literals $L \subseteq Lit(T)$, if $L^+ \subseteq M$ and $L^- \cap M = \emptyset$.

A set of atoms $M \subseteq At(T)$ is a *model* of a theory T if, for each clause $c \in T$, $c \cap M \neq \emptyset$ or $c^- \cap (At(T) \setminus M) \neq \emptyset$. A model M of a theory T is *minimal* if no proper subset of M is a model of T.

Let T be a CNF theory and let $L \subseteq Lit(T)$ be a set of literals. We define a theory T_L as follows:

$$T_L = \{c' : \text{there is } c \in T \text{ such that } c' = c - \overline{L} \text{ and } c \cap L = \emptyset\}.$$

Thus, to obtain T_L we remove from T all clauses implied by L (that is, clauses $c \in T$ such that $c \cap L \neq \emptyset$), and resolving each remaining clause with literals in L, that is, removing from it all literals in $\overline{L}$. It may happen that T_L contains the empty clause (is contradictory) or is empty (is a tautology). The theory T_L has the following important properties.

Lemma 1
Let T be a CNF theory and $L \subseteq Lit(T)$. For every $X \subseteq At(T)$ that is consistent with L:

1. X is a model of T if and only if $X - L^+$ is a model of T_L.
2. If X is a minimal model of T, then $X - L^+$ is a minimal model of T_L.
3. If $L^+ = \emptyset$ then X is a minimal model of T if and only if X is a minimal model of T_L.

Proof
Let T' be the set of clauses in T that contain a literal from L (in the proof we view clauses as sets of their literals). Clearly, when constructing T_L, we remove clauses in T' from T. Since X is consistent with L, X satisfies all clauses in T'. Thus, X is a model of T if and only if X is a model of $T'' = T - T'$.

Next, we note that every clause c in T'' is of the form $c' \cup c''$, where $c' \in T_L$ and c'' consists of literals of the form $\overline{\omega}$, for some $\omega \in L$. Moreover, every clause $c' \in T_L$ appears at least once as part of such representation of a clause c from T.

Since X is consistent with L, X is a model of a clause $c \in T''$ if and only if X is a model of c'. Since c' contains no literals from L nor their duals, X is a model of c' if and only $X - L^+$ is a model of c'. It follows that X is a model of T'' (and so, of T) if and only if $X - L^+$ is a model of T_L.

To prove part (2) of the assertion, we observe first that $X - L^+$ is a model of T_L (by part (1)). Let us consider $Y \subseteq X - L^+$ such that Y is a model of T_L. Clearly, $Y \cup L^+$ is consistent with L and $Y = (Y \cup L^+) - L^+$. Hence, it follows by (1) that $Y \cup L^+$ is a model of T. Since X is consistent with L, $Y \cup L^+ \subseteq X$. By the minimality of X, $Y \cup L^+ = X$ and, as $Y \cap L^+ = \emptyset$, we obtain that $Y = X - L^+$. Thus, $X - L^+$ is a minimal model of T_L.

To prove (3), we only need to show that if X is a minimal model of T_L then X is a minimal model of T (the other implication follows from (2)). By (1), X is model of T. Let $Y \subseteq X$ be also a model of T. Clearly, Y is consistent with L. Thus, again by (1), Y is a model of T_L. By the minimality of X, $Y = X$ and X is a minimal model of T. □

Let T be a CNF theory. A family $\mathcal{A}$ of subsets of $Lit(T)$ *covers* all minimal models of T, or is a *cover* for T, if $\mathcal{A} \neq \emptyset$, $\emptyset \notin \mathcal{A}$ and if every minimal model of T is consistent with at least one set $A \in \mathcal{A}$. A *cover function* is a function which, to every CNF theory T such that $At(T) \neq \emptyset$ assigns a cover of T. The family $\mathcal{A} = \{\{a\}, \{\overline{a}\}\}$, where a is an atom of T, is an example of a cover for T. A function which, to every CNF theory T such that $At(T) \neq \emptyset$, assigns $\{\{a\}, \{\overline{a}\}\}$, for some atom $a \in At(T)$, is an example of a cover function.

When processing CNF theories, it is often useful to simplify their structure without changing their logical properties. Let σ be a function assigning to each CNF theory T another CNF theory, $\sigma(T)$, which is equivalent to T, satisfies $At(\sigma(T)) \subseteq At(T)$ and which, in some sense, is simpler than T. We call each such function a *simplifying* function. For now, we leave the nature of the simplifications encoded by σ open.

We are now in a position to describe the algorithm we promised at the beginning of this section. That algorithm computes a superset of the set of all minimal models of an input CNF theory T. It is parameterized with a cover function ρ and a simplifying function σ. That is, different choices for ρ and σ specify different instances of the algorithm. We call this algorithm min^+ and describe it in Figure 1. To be precise, the notation should explicitly refer to the functions ρ and σ that determine min^+. We omit these references to keep the notation simple. The functions ρ and σ will always be clear from the context.

The algorithm min^+ is fundamental to our approach. We derive from it algorithms for the three main tasks of interest to us: computing minimal models, stable models and answer sets.

The input parameters of min^+ are CNF theories T and S, and a set of literals L. We require that $L \subseteq Lit(T)$ and $S = \sigma(T_L)$. We will refer to these two conditions as the *preconditions* for min^+. The input parameter S is determined by the two other parameters T and L (through the preconditions on T, S and L). We choose to specify it explicitly as that simplifies the description and the analysis of the algorithm.

The output of the algorithm $min^+(T, S, L)$ is a family $\mathcal{M}^+(T, L)$ of sets that contains all minimal models of T that are consistent with L.

The algorithm min^+ and the implementations of the cover function ρ and a simplifying function σ that are used in min^+ assume a standard linked-list representation of CNF theories. Specifically, an input CNF theory T is a doubly-linked list of its clauses, where each clause c in T is a doubly-linked list of its literals. The total size of such a representation of a CNF theory T is $O(m)$. In addition, for each literal $\omega \in Lit(T)$, we have a linked list $C(\omega)$ consisting of all clauses in T that contain ω as a literal. More precisely, for each clause c containing ω, the list $C(\omega)$ contains a pointer to the location of c in T and a pointer to the location of ω on the list c. These lists can be created from the linked list T in time that is linear in the size of T. Since we assume that clauses do not contain multiple occurrences of the same literal, we assume that the same property holds for linked lists that represent them.

Clearly, the recursive call in the line (9) is legal as $SA = \sigma(T_{L \cup A})$. Moreover,

```
min+(T, S, L)
    % T and S are CNF theories, L is a set of literals
    % T, S and L satisfy the preconditions: L ⊆ Lit(T), and S = σ(T_L)
1   if S does not contain an empty clause then
2     if S = ∅ then
3       M := L+; output(M)
4     else
5       A := ρ(S);
6       for every A ∈ A do
7         SA' := T_{L∪A};
8         SA := σ(SA');
9         min+(T, SA, L ∪ A)
10      end of for
11    end of else
12  end of min+.
```

Fig. 1. Algorithm min^+

since ρ is a cover function, for every $A \in \rho(S)$, $|A| \geq 1$. Thus, $|At(SA)| < |At(S)|$ and the algorithm terminates. The next lemma establishes the key property of the output produced by the algorithm min^+.

Lemma 2

Let ρ be a cover function and σ be a simplifying function. For every CNF theory T and a set of literals $L \subseteq Lit(T)$, if X is a minimal model of T consistent with L, then X is among the sets returned by $min^+(T, S, L)$, where $S = \sigma(T_L)$.

Proof

We prove the assertion proceeding by induction on $|At(S)|$. Let us assume that $|At(S)| = 0$ and that X is a minimal model of T. By Lemma 1, $X - L^+$ is a minimal model of T_L and, consequently, also of S ($S = \sigma(T_L)$ and so, S and T_L have the same models). It follows that S is consistent and, therefore, contains no empty clause. Since $|At(S)| = 0$, $S = \emptyset$. Consequently, $X - L^+ = \emptyset$ (as $X - L^+$ is a minimal model of S). Hence, $X \subseteq L^+$. Furthermore, since X is consistent with L, $L^+ \subseteq X$. Thus, $X = L^+$. Finally, since S is empty, the program enters line (3) and outputs X, as $X = L^+$.

For the inductive step, let us assume that $|At(S)| > 0$ and that X is a minimal model of T consistent with L. By Lemma 1, $X - L^+$ is a minimal model of T_L and, consequently, a minimal model of S. Since $\mathcal{A}$, computed in line (5), is a cover for S, there is $A \in \mathcal{A}$ such that $X - L^+$ is consistent with A. Clearly, $At(L) \cap At(A) = \emptyset$. Thus, X is consistent with $L \cup A$. By the induction hypothesis (the parameters T, SA and $L \cup A$ satisfy the preconditions for the algorithm min^+ and $|At(S)| > |At(SA)|$), the call $min^+(T, SA, L \cup A)$, within loop (6), returns the set X. □

Corollary 1

Let T be a CNF theory. The family $\mathcal{M}^+(T, \emptyset)$ of sets that are returned by $min^+(T, S, \emptyset)$, where $S = \sigma(T)$, contains all minimal models of T.

We will now study the performance of the algorithm min^+. We start with the following observation concerning computing theories SA' in line (7) of the algorithm min^+.

Lemma 3
There is an algorithm which, given a linked-list representation of a CNF theory S and a set A of literals such that $A \subseteq Lit(S)$, constructs a linked-list representation of the theory SA' in linear time in the size of S.

Proof
It is clear that with the data structures that we described above, we can eliminate from the list S all clauses containing literals in A by traversing lists $C(\omega)$, $\omega \in A$, and by deleting each clause we encounter in this way. Since S is a doubly-linked list, each deletion takes constant time, and the overall task takes linear time in the size of S. Similarly, also in linear time in the size of the list S, we can remove literals in the set $\overline{A}$ from each clause that remains on S. □

Next, we will analyze the recursive structure of the algorithm min^+. Let ρ be a cover function and σ a simplifying function. For a CNF theory T we define a labeled tree $\mathcal{T}_T$ inductively as follows. If T contains the empty clause or if $T = \emptyset$ (T is a tautology), $\mathcal{T}_T$ consists of a single node labeled with $\sigma(T)$. Otherwise, we form a new node, label it with $\sigma(T)$ and make it the parent of all trees $\mathcal{T}_{T'_A}$, where $T' = \sigma(T)$, $A \in \rho(T')$. For every $A \in \rho(T)$ we have $|At(T'_A)| < |At(T')| \leq |At(T)|$. Thus, the tree $\mathcal{T}_T$ is well defined. We denote the set of leaves of the tree $\mathcal{T}_T$ by $L(\mathcal{T}_T)$. As the algorithm min^+, $\mathcal{T}_T$ depends on functions σ and ρ, too. We drop the references to these functions to keep the notation simple.

It is clear that for every CNF theory T and for every set of literals $L \subseteq Lit(T)$, the tree $\mathcal{T}_S$, where $S = T_L$, is precisely the tree of recursive calls to min^+ made by the top-level call $min^+(T, \sigma(S), L)$. In particular, the tree $\mathcal{T}_T$ describes the structure of the execution of the call $min^+(T, \sigma(T), \emptyset)$.

We use the tree $\mathcal{T}_T$ to estimate the running time of the algorithm min^+. We say that a cover function ρ is *splitting* if for every theory T, such that $|At(T)| \geq 2$, it returns a cover with at least two elements. We have the following result.

Lemma 4
Let ρ be a splitting cover function and σ a simplifying function. Let t be an integer function such that $t(k) = \Omega(k)$ and the functions ρ and σ can be computed in time $O(t(k))$ on input theories of size k. Then the running time of the algorithm min^+ on input $(T, \sigma(T), \emptyset)$, where T is a CNF theory, is $O(|L(\mathcal{T}_T)|t(m))$.

Proof
Let T be a CNF theory and let s be the number of nodes in the tree $\mathcal{T}_T$. Then, the number of recursive calls of the algorithm min^+ is also equal to s. Clearly, the total time needed for lines (1)-(6) over all recursive calls to min^+ is $O(s \cdot t(m))$ (in the case of line (6) we only count the time needed to control the loop and not to execute its content). Indeed, in each recursive call the size of the theories considered is bounded by m, the size of the theory T.

We charge each execution of the code in lines (7) and (8) to the recursive call to min^+ that immediately follows. By Lemma 3, line (7) takes time $O(m)$ and line (8) takes time $O(t(m))$ (again, by the fact that the sizes of theories SA and SA' are bounded by the size of T, that is, by m). Thus, the total time needed for these instructions over all recursive calls to min^+ is also $O(s \cdot t(m))$.

It follows that, the running time of the algorithm min^+ is $O(st(m))$. Since ρ is splitting, every node in the tree $\mathcal{T}_T$, other than leaves and their parents, has at least two children. Consequently, $s = O(|L(\mathcal{T}_T)|)$ and the assertion follows. □

We also note that only those recursive calls to min^+ that correspond to leaves of $\mathcal{T}_T$ produce output. Thus, Corollary 1 imply the following bound on the number of minimal models of a CNF theory T.

Lemma 5
Let T be a CNF theory. The number of minimal models of T is at most $|L(\mathcal{T}_T)|$.

In order to use Lemmas 4 and 5 we need a method to estimate the number of leaves in rooted trees. Let $\mathcal{T}$ be a rooted tree and let $L(\mathcal{T})$ be the set of leaves in $\mathcal{T}$. For a node x in $\mathcal{T}$, we denote by $C(x)$ the set of *directed* edges in $\mathcal{T}$ that link x with its children. For a leaf w of $\mathcal{T}$, we denote by $P(w)$ the set of *directed* edges on the unique path in $\mathcal{T}$ from the root of $\mathcal{T}$ to the leaf w. The following observation was shown in (Kullmann 1999).

Lemma 6
(Kullmann 1999) Let p be a function assigning positive real numbers to edges of a rooted tree $\mathcal{T}$ such that for every internal node x in $\mathcal{T}$, $\sum_{e \in C(x)} p(e) = 1$. Then,

$$|L(\mathcal{T})| \leq \max_{w \in L(\mathcal{T})} \Big(\prod_{e \in P(w)} p(e) \Big)^{-1}.$$

For some particular cover functions, Lemma 6 implies specific bounds on the number of leaves in the tree $\mathcal{T}_T$. Let μ be a function that assigns to every CNF theory T a real number $\mu(T)$ such that $0 \leq \mu(T) \leq |At(T)|$. We call each such function a *measure*. Given a measure μ, a simplifying function σ is *μ-compatible* if for every CNF theory S, $\mu(\sigma(S)) \leq \mu(S)$. Similarly, we say that a cover function ρ is *μ-compatible* if for every CNF theory S such that $At(S) \neq \emptyset$ and for every $A \in \rho(S)$, $\mu(S) - \mu(S_A) > 0$. We denote the quantity $\mu(S) - \mu(S_A)$ by $\Delta(S, S_A)$.

Let S be a CNF theory such that $At(S) \neq \emptyset$. Let μ be a measure and let ρ be a cover function that is μ-compatible. Since for every $A \in \rho(A)$, $\Delta(S, S_A) > 0$, there is a unique real number $\tau \geq 1$ satisfying the equation

$$\sum_{A \in \rho(S)} \tau^{-\Delta(S, S_A)} = 1. \tag{1}$$

Indeed, for $\tau \geq 1$ the left hand side of the equation (1) is a strictly decreasing continuous function of τ. Furthermore, its value for $\tau = 1$ is at least 1 (as $\rho(S) \neq \emptyset$) and it approaches 0 when τ tends to infinity. We denote the number $\tau \geq 1$ satisfying (1) by τ_S (we drop references to μ and ρ, on which τ_S also depends, to keep the notation simple). We say that ρ is *μ-bounded* by a real number τ_0 if for every CNF theory S with $|At(S)| \geq 1$, $\tau_S \leq \tau_0$. We have the following result — a corollary to Lemma 6.

Lemma 7
Let T be a CNF theory and let μ be a measure. For every μ-compatible simplifying function σ and for every μ-compatible cover function ρ that is μ-bounded by τ_0,

$$|L(\mathcal{T}_T)| \leq \tau_0^{|At(T)|}. \tag{2}$$

Proof
Let $e = (x, y)$ be an edge in $\mathcal{T}_T$ and let S and S' be CNF theories that label x and y, respectively. Since x is not a leaf in $\mathcal{T}_T$, $At(S) \neq \emptyset$. Thus, ρ is defined for S. Moreover, by the definition of $\mathcal{T}_T$ it follows that there is an element $A \in \rho(S)$ such that $S' = \sigma(S_A)$. We define $D(e) = \mu(S) - \mu(S')$. Since σ is μ-compatible, $\mu(S') \leq \mu(S_A)$. Thus, $D(e) \geq \mu(S) - \mu(S_A)$. Since ρ is μ-compatible, $D(e) > 0$.

Let us now set $p(e) = \tau_S^{-\Delta(S,S_A)}$, where τ_S is the root of the equation (1). Since, ρ is μ-bounded by τ_0, we have $p(e)^{-1} = \tau_S^{\Delta(S,S_A)} \leq \tau_0^{\Delta(S,S_A)} \leq \tau_0^{D(e)}$ (we recall that $\tau_S \geq 1$).

Clearly, for every leaf $w \in \mathcal{T}_T$ we have

$$\sum_{e \in P(w)} D(e) = \mu(T') - \mu(W) \leq \mu(T) \leq |At(T)|,$$

where $T' = \sigma(T)$ and W is the theory that labels w. Thus,

$$(\prod_{e \in P(w)} p(e))^{-1} \leq \prod_{e \in P(w)} \tau_0^{D(e)} = \tau_0^{\sum_{e \in P(w)} D(e)} \leq \tau_0^{|At(T)|}.$$

The function p satisfies the assumptions of Lemma 6. Consequently, the assertion follows. □

Results of this section show that in order to get good performance bounds for the algorithm min^+ and good bounds on the number of minimal models of a CNF theory one needs a measure μ and a splitting cover function ρ that is μ-compatible and μ-bounded by τ_0, where τ_0 is as small as possible.

To estimate roots τ_S of specific equations of type (1), which we need to do in order to bound all of them from above and determine τ_0, we will later use the following straightforward observation.

Lemma 8
Let μ be a measure, ρ a μ-compatible cover function and S a CNF theory with $|At(S)| \geq 1$. If for every $A \in \rho(S)$, $k_{S,A}$ is a positive real such that $\Delta(S, S_A) \geq k_{S,A}$, then $\tau_S \leq \tau'_S$, where τ'_S is the root of the equation

$$\sum_{A \in \rho(S)} \tau^{-k_{S,A}} = 1. \tag{3}$$

4 Computing minimal models, stable models and answer sets — a general case

We will now derive from the algorithm min^+ algorithms for computing minimal models of CNF theories, stable models of normal programs and answer sets of disjunctive programs. In this section, we do not assume any syntactic restrictions.

We start with the problem of computing minimal models of a CNF theory T. Let *test_min* be an algorithm which, for a given CNF theory T and a set of atoms $M \subseteq At(T)$ returns the boolean value **true** if M is a minimal model of T, and returns **false**, otherwise.

We now modify the algorithm min^+ by replacing each occurrence of the command **output**(M) (in line (3)), with the command

if $test_min(T, M)$ **then output**(M).

We denote the resulting algorithm by *min_mod* (we assume the same preconditions on *min_mod* as in the case of min^+). Since all minimal models of T that are consistent with L belong to $\mathcal{M}^+(T, L)$ (the output of $min^+(T, \sigma(T_L), L)$), it is clear that the algorithm $min_mod(T, \sigma(T_L), L)$ returns all minimal models of T that are consistent with L and nothing else.

Computation of stable models and answer sets of logic programs follows a similar pattern. First, let us recall that we can associate with a disjunctive logic program P (therefore, also with every normal logic program P) its propositional counterpart, a CNF theory $T(P)$ consisting of clauses of P but interpreted in propositional logic and rewritten into CNF. Specifically, to obtain $T(P)$ we replace each disjunctive program clause

$$c_1 \vee \ldots \vee c_p \leftarrow a_1, \ldots, a_r, \mathbf{not}(b_1), \ldots, \mathbf{not}(b_s)$$

in P with a CNF clause

$$\neg a_1 \vee \ldots \vee \neg a_r \vee b_1 \vee \ldots \vee b_s \vee c_1 \vee \ldots \vee c_p.$$

It is well known that stable models (answer sets) of (disjunctive) logic program P are minimal models of $T(P)$ (Marek and Truszczyński 1993).

Let us assume that $test_stb(P, M)$ and $test_anset(P, M)$ are algorithms to check whether a set of atoms M is a stable model and an answer set, respectively, of a program P.

To compute stable models of a logic program P that are consistent with a set of literals L, we first compute the CNF theory $T(P)$. Next, we run on the triple $T(P)$, $\sigma(T(P)_L)$ and L, the algorithm min^+ modified similarly as before (we note that the triple $(T(P), \sigma(T(P)_L), L)$ satisfies the preconditions of min^+). Namely, we replace the command **output**(M) (line (3)) with the command

if $test_stb(P, M)$ **then output**(M).

The effect of the change is that we output only those sets in $\mathcal{M}^+(T(P), L)$, which are stable models of P. Since every stable model of P is a minimal model of $T(P)$, it is an element of $\mathcal{M}^+(T(P), L)$. Thus, this modified algorithm, we will refer to it as *stb_mod*, indeed outputs all stable models of P consistent with L and nothing else.

In the same way, we construct an algorithm *ans_set* computing answer sets of disjunctive programs. The only difference is that we use the algorithm *test_anset* in place of *test_stb* to decide whether to output a set.

We will now analyze the performance of the algorithms we just described. The following observation is evident.

Lemma 9
Let ρ be a splitting cover function and let

1. the worst-case running time of the algorithm min^+ be $O(t_1(n,m))$, for some integer function t_1, and
2. the worst-case running time of the algorithm *test_min* (*test_stb* or *test_anset*, depending on the problem) be $O(t_2(n,m))$, for some integer function t_2.

Then the running time of the algorithms *min_mod*, *stb_mod* and *ans_set* (in the worst case) is $O(t_1(n,m)+\gamma t_2(n,m))$, where $\gamma = |L(\mathcal{T}_T)|$ or $|L(\mathcal{T}_{T(P)})|$, depending on the problem, and T and P are an input CNF theory or an input (disjunctive) program, respectively.

Proof
Clearly, the running time of the algorithm *test_min* (and, similarly, *test_stb* and *test_anset*) is the sum of the running times of the algorithm min^+ and of all the calls to the algorithm *min_mod* (*stb_mod* and *ans_set*, respectively). The number of calls to the algorithm *min_mod* (*stb_mod* and *ans_set*, respectively) is equal to the number of nodes in the tree $\mathcal{T}_T$ ($\mathcal{T}_{T(P)}$, respectively). Since the cover function ρ is splitting, that number is $O(\gamma)$. Thus, the assertion follows. □

5 2-CNF theories, 2-programs, disjunctive 2-programs

The performance of the algorithms *min_mod*, *stb_mod* and *ans_set* depends on the performance of the algorithm min^+ and on the performance of the algorithms *test_min*, *test_stb* and *test_anset*. The performance of the algorithm min^+ depends, in turn, on the performance of the implementations of the underlying cover function ρ and the simplifying function σ.

We note that if T is a 2-CNF theory, then throughout the execution of the algorithm min^+ we only encounter 2-CNF theories (theories S_A are 2-CNF theories if S is a 2-CNF theory). Thus, it is enough to define and implement a simplifying function σ and a cover function ρ for 2-CNF theories only. We define $\sigma(T) = T$, for every 2-CNF theory T (we choose the identity function for σ) and we have the following result concerning ρ.

Lemma 10
Let a simplifying function σ be the identity function. There is a splitting cover function ρ for 2-CNF theories that can be implemented to run in time $O(m)$ and such that for every 2-CNF theory T, $|L(\mathcal{T}_T)| \leq 1.4422..^n$.

Proof
We recall that we only consider CNF theories that do not contain clauses with multiple occurrences of the same literal or occurrences of both a literal and its dual. Thus, theories we consider here contain no clauses of the form $\gamma \vee \overline{\gamma}$, $\gamma \vee \gamma$, where γ is a literal.

To define a splitting function ρ for a 2-CNF theory S with $At(S) \neq \emptyset$, we will

consider several cases depending on the properties of S. In each of them we assume that situations covered by the ones considered before are excluded.

Case 1. $|At(S)| = 1$. Let us assume that $At(S) = \{x\}$. In this case, $S = \{x\}$ or $S = \{\overline{x}\}$ (in this proof, we view clauses as disjunctions of literals). We define $A_1 = \{x\}$ or $A_1 = \{\overline{x}\}$, respectively. Clearly, $\{A_1\}$ is a cover for S. We set $\rho(S) = \{A_1\}$.

Case 2. There is a literal ω such that the unit clause ω belongs to S. Let y be an atom in $At(S)$ such that y is not the atom in ω (such an atom exists, as $|At(S)| \geq 2$ now). We define $A_1 = \{\omega, y\}$ and $A_2 = \{\omega, \overline{y}\}$. Clearly, $\{A_1, A_2\}$ is a cover for S. Indeed, if M is a minimal model of S, then M satisfies ω and either M satisfies y or M satisfies $\overline{y}$. We set $\rho(S) = \{A_1, A_2\}$. We also observe that, since y does not appear in ω, $|A_i| = 2$, $i = 1, 2$.

Case 3. There is an atom x such that all its occurrences in clauses of S are negative. Let y be any *other* atom in S (as we argued above, such y exists). We set $A_1 = \{\overline{x}, y\}$ and $A_2 = \{\overline{x}, \overline{y}\}$. Let M be a minimal model of S. Then M satisfies $\overline{x}$ (otherwise, $M' = M - \{x\}$ would be a model of S, a contradiction with the minimality of M). Since M satisfies either y or $\overline{y}$, $\{A_1, A_2\}$ is a cover for S. We define $\rho(S) = \{A_1, A_2\}$. Since $x \neq y$, we have $|A_i| = 2$, $i = 1, 2$.

Case 4. There is a clause $\overline{x} \vee \omega$ in S. Since we assume now that Case 3 does not hold, there is also a clause $x \vee \beta$ in S. In this case, we define $A_1 = \{x, \omega\}$ and $A_2 = \{\overline{x}, \beta\}$. It is easy to verify that $\{A_1, A_2\}$ is a cover for S and we define $\rho(S) = \{A_1, A_2\}$. Since x does not appear in ω and β (we assume that S does not contain clauses of the form $\gamma \vee \gamma$ and $\gamma \vee \overline{\gamma}$, where γ is a literal), we also have $|A_i| = 2$, $i = 1, 2$.

Case 5. All clauses in S are of the form $x \vee y$, where x and y are different atoms.

(a) There is an atom, say x, that appears in exactly one clause, say $x \vee y$. Let us define $A_1 = \{x, \overline{y}\}$ and $A_2 = \{\overline{x}, y\}$, and let M be a minimal model of S. If $x \in M$ then $y \notin M$. Indeed, otherwise $M - \{x\}$ would be a model of S (as x appears only in the clause $x \vee y$ in S). That would contradict the minimality of M. Thus, if $x \in M$, M is consistent with A_1. If $x \notin M$ then, since M is a model of $x \vee y$, $y \in M$. Thus, in this case, M is consistent with A_2. It follows that $\{A_1, A_2\}$ is a cover for S and we define $\rho(S) = \{A_1, A_2\}$. Moreover, we also have that $|A_i| = 2$, $i = 1, 2$.

(b) There is an atom x that appears in at least three different clauses, say $x \vee y_i$, $i = 1, 2, 3$. In this case, we set $A_1 = \{x\}$ and $A_2 = \{\overline{x}, y_1, y_2, y_3\}$. It is evident that $\{A_1, A_2\}$ is a cover for S and we set $\rho(S) = \{A_1, A_2\}$. We also note that $|A_1| = 1$ and $|A_2| = 4$.

(c) Every atom appears in exactly two clauses. Let w be an arbitrary atom in S. Let $w \vee u$ and $w \vee v$ be the two clauses in S that contain w. Both u and v also appear in exactly two clauses in S. Let $u \vee u'$ be the clause other than $w \vee u$ (that is, $u' \neq w$) and $v \vee v'$ be the clause other than $w \vee v$ (that is, $v' \neq w$). We set $A_1 = \{\overline{u}, u', w\}$, $A_2 = \{\overline{v}, v', w\}$ and $A_3 = \{\overline{w}, u, v\}$. By the construction, $|A_i| = 3$, $i = 1, 2, 3$. Moreover, the family $\{A_1, A_2, A_3\}$ is a cover for S. Indeed, let M be a minimal model of S. If $u, v \in M$, then $w \notin M$, since M is a minimal model of S and $w \vee u$ and $w \vee v$ are the only clauses in S containing w. If $u \notin M$, then both w and u' are in M. If $v \notin M$, then w and v' are in M. Thus, we define $\rho(S) = \{A_1, A_2, A_3\}$.

Cases 1 - 5 exhaust all possibilities for a 2-CNF theory S such that $At(S) \neq \emptyset$. Thus, the definition of ρ is complete. We argued in each case that $\rho(S)$ is a cover. Therefore ρ is a cover function. Moreover, in each case when $|At(S)| \geq 2$, $|\rho(S)| \geq 2$. Thus, the cover function ρ is splitting.

The definition of ρ implies a simple algorithm for computing $\rho(S)$. One needs to identify the first case that applies (it can be accomplished in linear time in the size of S given a linked-list representation we discussed in Section 3) and output the cover constructed in that case (it can be done in constant time). Thus, the overall algorithm can be implemented to run in linear time.

To estimate the number of leaves in the tree $\mathcal{T}_T$, where T is a 2-CNF theory, we will use Lemma 7. To this end, for each 2-CNF theory S we define $\mu(S) = |At(S)|$. It is evident that $0 \leq \mu(S) \leq |At(S)|$. Thus, μ is a measure.

Since for every 2-CNF theory S, $\sigma(S) = S$, σ is μ-compatible. Next, it follows directly from the definition of ρ that for every 2-CNF theory S with $At(S) \neq \emptyset$ and for every $A \in \rho(S)$, $\mu(S) - \mu(S_A) > 0$. Thus, ρ is also μ-compatible. We will now show that ρ is μ-bounded by 1.4422.. . In order to do it, for every 2-CNF theory S with $|At(S)| \geq 1$ we estimate τ_S, the unique positive root of the equation (1).

Let S be 2-CNF theory such that $|At(S)| \geq 1$. In Case 1, $\rho(S)$ consists of exactly one set, say A_1, and $S_{A_1} = \emptyset$. Thus, $\mu(S) = 1$, $\mu(S_{A_1}) = 0$ and $\Delta(S, S_{A_1}) = 1$. The equation (1) becomes $\tau = 1$ and $\tau_S = 1$.

In Cases 2 - 4 and 5a, $\rho(S)$ consists of two sets, A_1 and A_2, and $\mu(S_{A_i}) \leq \mu(S) - 2$, $i = 1, 2$. In other words, $\Delta(S, S_{A_i}) \geq 2$, $i = 1, 2$. In each such case, Lemma 8 implies that $\tau_S \leq \tau_1$, where τ_1 is the root of the equation $2\tau^{-2} = 1$.

In Case 5b, S has two children in $\mathcal{T}_T$, S_{A_i}, $i = 1, 2$. Moreover, $\mu(S_{A_1}) \leq \mu(S) - 1$ and $\mu(S_{A_2}) \leq \mu(S) - 4$. Consequently, $\Delta(S, S_{A_1}) \geq 1$ and $\Delta(S, S_{A_2}) \geq 4$. In this case, Lemma 8 implies that $\tau_S \leq \tau_2$, where τ_2 is the root of the equation $\tau^{-1} + \tau^{-4} = 1$.

Finally, in Case 5c, S has three children in $\mathcal{T}_T$, S_{A_i}, $i = 1, 2, 3$, and $\mu(S_{A_i}) \leq \mu(S) - 3$, $i = 1, 2, 3$. In other words, $\Delta(S, S_{A_i}) \geq 3$, $i = 1, 2, 3$. In this case, $\tau_S \leq \tau_3$, where τ_3 is the root of the equation $3\tau^{-3} = 1$.

Since $\tau_1 = 1.4142..$, $\tau_2 = 1.3802..$ and $\tau_3 = 1.4422..$, it follows by Lemma 7 that $|L(\mathcal{T}_T)| \leq (1.4422..)^n$. Thus, the assertion follows. □

Proof of Theorem 2. In Section 3 we noted that for every (disjunctive) 2-program P, stable models (answer sets) of P are minimal models of the theory $T(P)$. Thus, Theorem 2 follows from Lemmas 10 and 5.

In addition, Lemmas 10, 3 and 4 imply the following corollary.

Corollary 2
There is an implementation of the algorithm min^+ that, for 2-CNF theories, runs in time $O(m1.4422..^n)$.

To derive Theorem 1 we will need one more auxiliary fact concerning testing whether a set of atoms is a minimal model of a 2-CNF theory.

Proposition 1

Let T be a 2-CNF theory and $M \subseteq At(T)$. There is a linear-time algorithm to test whether M is a minimal model of T.

Proof
First, we test whether M is a model of T. To this end, we check whether every clause in T has a literal that is true in M (a literal of the form a, where $a \in M$, or a literal of the form $\overline{a}$, where $a \notin M$). That task can be accomplished in linear time in the size of T. If M is not a model of T, it is not a minimal model. So, from now on we will assume that M is a model of T.

We now define $L = \{\overline{a}\colon a \in At(T) - M\}$. By Lemma 1(3), M is a minimal model of T if and only if M is a minimal model of T_L.

Let c be a clause of T_L. By Lemma 1(1), M is a model of c. Let us assume that c consists of negated atoms only. Then, c contains a literal $\overline{a}$, where a is an atom, and M satisfies $\overline{a}$. On the other hand, since $c \in T_L$, it follows from the way T_L is defined that $a \in M$. Thus, we get a contradiction. Consequently, every clause in T_L is of the form a, $a \vee b$ or $a \vee \overline{b}$. In particular, it follows that $At(T_L)$ is a model of T_L. By the construction of T_L, $At(T_L) \subseteq M$. Thus, if $At(T_L) \neq M$, M is not a minimal model of T_L. Therefore, from now on we can assume that $At(T_L) = M$.

We form a directed graph G by using atoms in $At(T_L)(= M)$ as its vertices and by connecting vertices a and b with a directed edge (a, b) if and only if $\overline{a} \vee b$ is a clause in T_L. Strongly connected components in the graph G consist of atoms that must have the same truth value in every model M' of T_L (either all atoms of a strongly connected component of G are in M' or none of them is).

By a *0-rank* strongly connected component of G we mean a strongly connected component S of G such that there is no edge (a, b) in G with $a \notin S$ and $b \in S$. One can show that M is a minimal model of T_L if and only if every 0-rank strongly connected component in G contains at least one positive clause of T_L.

Computing strongly connected components and 0-rank strongly connected components can be done in linear time. Similarly, one can verify in linear time whether every 0-rank component contains a positive clause from T_L. Thus, the method described in the proof can be implemented to run in linear time. □

Proof of Theorem 1. It is well known that testing whether a set of atoms M is a stable model of a logic program can be done in linear time. Testing whether a set of atoms M is a minimal model of a 2-CNF theory can be done in linear time according to Proposition 1. To test whether a set of atoms M is an answer set of a disjunctive 2-program one needs to test whether M is a minimal model of the reduct P^M or, equivalently, whether M is a minimal model of a 2-CNF theory $T(P^M)$. Thus, by Proposition 1, also this task can be accomplished in linear time. Consequently Theorem 1 follows from Lemma 9 and Corollary 2.

6 3-CNF theories, 3-programs, disjunctive 3-programs

In this section, we will prove Theorems 3, 4 and 5. As with 2-CNF theories, the first step is to specify functions σ and ρ. Let T be a 3-CNF theory. We define $\sigma(T)$

to be the theory obtained by eliminating multiple occurrences of clauses and all 3-clauses that are subsumed by 2-clauses in T.

Lemma 11
Let T be a 3-CNF theory. There is an algorithm that computes $\sigma(T)$ and runs in linear time in the size of T (assuming a linked-list representation of T).

Proof
We first create a linked list Q of clauses that contains for each clause c in T all lists obtained from c by permuting its elements. With each permutation of c, we store a pointer to c on the list (representing) T. The list Q has size at most six times larger than the size of T.

Next, we sort Q lexicographically. This task can be accomplished in linear time by the radix sort algorithm (Aho et al. 1974). Clearly, if T contains $r \geq 2$ occurrences of a clause c, then for every permutation c' of the literals in c, Q will contain a contiguous segment of r occurrences of c'. Conversely, each contiguous segment of identical elements on the list Q indicates a clause that appears in T multiple times. Thus, we can identify all clauses with multiple occurrences in T in a single pass through Q (we recall that we maintain pointers from clauses in Q to their original counterparts in T) and, then, delete duplicates.

Similarly, a 3-clause $c \in T$ is subsumed by a 2-clause $d \in T$ if and only if some permutation d' of d is a prefix of some permutation c' of c and so, d' is an immediate predecessor of c' on Q. As before, all such 3-clauses can be identified in a single pass through Q and then deleted from T.

Since we maintain T as a doubly-linked list, each deletion can be performed in linear time. Consequently, the whole process takes linear time in the size of T. □

We will next specify an appropriate cover function ρ. Due to the choice of σ, it suffices to define $\rho(T)$ for every 3-CNF theory T such that $At(T) \neq \emptyset$ and T contains no 3-clauses subsumed by 2-clauses in T, as that is enough to determine the tree $\mathcal{T}_T$. We have the following result.

Lemma 12
There is a splitting cover function ρ defined for every 3-CNF theory T that contains no multiple clauses nor 3-clauses subsumed by 2-clauses in T, which can be implemented to run in linear time and such that for every 3-CNF theory T, $|L(\mathcal{T}_T)| \leq 1.6701..^n$.

We outline a proof of Lemma 12 in the next section and provide full details in the appendix.

Proof of Theorem 3. Theorem 3 follows directly from Lemmas 5 and 12.

Next, we note that Lemmas 3, 4 and 12 imply the following corollary.

Corollary 3
There is an implementation of the algorithm min^+ that, for 3-CNF theories, runs in time $O(m1.6701..^n)$.

Proof of Theorem 4. Since there is a linear-time algorithm to test whether a set of atoms is a stable model of a logic program, Theorem 4 follows from Corollary 3 and Lemma 9.

We will now prove Theorem 5. We focus on the case of minimal models of 3-CNF theories. The argument in the case of answer sets of disjunctive 3-programs is similar. We start with a simple result on testing minimality.

Proposition 2
Let f be an integer function and t an integer such that $t \geq 2$. If there is an algorithm that decides in time $O(f(m,n))$ whether a t-CNF theory T is satisfiable, then there is an algorithm that decides in time $O(|M|f(m+1,n))$ whether a set $M \subseteq At(T)$ is a minimal model of a t-CNF theory T.

Proof
Let $M = \{a_1, \ldots, a_k\}$. We define $L = \{\overline{x}\colon x \in At(T) - M\}$ and observe that M is a minimal model of T if and only if M is a model of T and none of t-CNF theories $T_L \cup \{\overline{a}_i\}$, $i = 1, \ldots, k$, is satisfiable. Thus, to decide if M is a minimal model of T, we first check if M is a model of T (in time $O(m)$) and then apply the algorithm checking satisfiability of $k = |M|$ t-CNF theories $T_L \cup \{\overline{a}_i\}$ of size $m+1$ each (in time $O(f(m+1,n))$ each). □

Satisfiability of 3-CNF theories can be decided in time $O(m1.481^n)$ (Dantsin et al. 2002). Thus, by Proposition 2, there is an an algorithm to decide whether a set $M \subseteq At(T)$ is a minimal model of a 3-CNF theory T that runs in time $O(|M|m \cdot 1.481^{|M|})$.

Proof of Theorem 5. We can assume that $n \geq 4$. Let β be a real number such that $0.6 \leq \beta < 1$ (we will specify β later). We will now estimate the running time of the algorithm *min_mod*, in which the procedure *test_min* is an implementation of the method described above. By the proof of Lemma 9, this running time is the sum of the running time of the algorithm min^+ and of the total time t_{min} needed to execute all calls to *test_min* throughout the execution of *min_mod*. By Corollary 3, the first component is $O(m1.6701..^n)$.

To estimate t_{min}, we split $\mathcal{M}^+(T,\emptyset)$ into two parts:

$$\mathcal{M}_1 = \{M \in \mathcal{M}^+(T,\emptyset)\colon |M| \geq \beta n\} \text{ and } \mathcal{M}_2 = \{M \in \mathcal{M}^+(T,\emptyset)\colon |M| < \beta n\}.$$

Clearly, the total time t_{min} needed to execute all calls to *test_min* throughout the execution of min^+ is:

$$t_{min} = O(\sum_{M \in \mathcal{M}_1} m|M|(1.481)^{|M|} + \sum_{M \in \mathcal{M}_2} m|M|(1.481)^{|M|}).$$

We have

$$\begin{aligned}
\sum_{M \in \mathcal{M}_1} m|M|(1.481)^{|M|} &\leq \sum_{i \geq \beta n} m\binom{n}{i} i(1.481)^i \\
&\leq m(n+1-\lceil \beta n \rceil)\lceil \beta n \rceil \binom{n}{\lceil \beta n \rceil}(1.481)^{\lceil \beta n \rceil} \\
&\leq \beta m n^2 \binom{n}{\lceil \beta n \rceil}(1.481)^{\lceil \beta n \rceil}.
\end{aligned}$$

The second inequality follows from that fact that for every i, $i \geq 0.6n$,

$$m\binom{n}{i} i(1.481)^i \geq m\binom{n}{i+1}(i+1)(1.481)^{i+1},$$

and from the observation that the number of terms in the sum is $n+1-\lceil \beta n \rceil$. The last inequality follows by an easy calculation from the assumptions that $n \geq 4$ and $\beta \geq 0.6$.

To estimate the second term, we note that $|\mathcal{M}_2| \leq |\mathcal{M}^+(T, \emptyset)| \leq (1.6701..)^n$ and, for every $M \in \mathcal{M}_2$, $|M| < \beta n$. Thus,

$$\sum_{M \in \mathcal{M}_2} m|M|(1.481)^{|M|} \leq (\beta mn)(1.6701..)^n(1.481)^{\beta n}.$$

Let us choose β to be the smallest $\beta' \geq 0 \cdot 6$ such that $\binom{n}{\lceil \beta' n \rceil} = O(1.6701..^n)$. One can verify that $\beta = 0.7907..$. For this β, we have

$$t_{min} = O(mn^2(1.6701..(1.481)^{\beta})^n) = O(mn^2 2.2782..^n),$$

which completes the proof of Theorem 5.

7 An outline of the proof of Lemma 12

To prove Lemma 12, we will follow a similar approach to that we used in the proof of Lemma 10. In the proof we define a cover function ρ and verify that it satisfies all the requirements of the lemma. Because of our choice of the simplifying function, it is enough to define ρ for every 3-CNF theory S such that $At(S) \neq \emptyset$ and S contains no multiple occurrences of clauses nor 3-clauses subsumed by 2-clauses of S.

To estimate the number of leaves in the tree $\mathcal{T}_T$, we will introduce a measure μ and show that σ is μ-compatible and that ρ is μ-compatible and μ-bounded by 1.6701.. . Specifically, for a 3-CNF theory S, we define $\mu(S) = n(S) - \alpha k(S)$, where $n(S) = |At(S)|$, $k(S)$ is the maximum number of 2-clauses in S with pairwise disjoint sets of atoms, and α is a constant such that $0 < \alpha < 1$. We will specify the constant α later. At this point we only note that for every $\alpha \in [0, 1]$ and for every 3-CNF theory S, $0 \leq \mu(S) \leq |At(S)|$. Thus, μ is indeed a measure (no matter what $\alpha \in [0, 1]$ we choose).

Let S be a 3-CNF theory such that $|At(S)| \geq 1$. If $S' = \sigma(S)$ then $|At(S)| \geq |At(S')|$. Moreover, since the sets of 2-clauses in S and S' are the same, $k(S) = k(S')$. Thus, $\mu(S) \geq \mu(S')$ and, consequently, σ is μ-compatible.

We will now outline the construction of $\rho(S)$, where S is a 3-CNF theory such that $At(S) \neq \emptyset$ and S contains no multiple occurrences of clauses nor 3-clauses subsumed by 2-clauses in S. We consider several cases depending on the structure of S. We design the cases so that every such 3-CNF theory S falls into exactly one of them. Moreover, we design these cases so that one can decide which case applies to S in linear time in the size of S. In each case, for a 3-CNF theory S we describe a specific cover for S and use it as the value of $\rho(S)$. In each case, it is clear that $\rho(S)$ can be output in constant time. Consequently, it follows that computing $\rho(S)$ can be accomplished in linear time. Moreover, in each case when $|At(S)| \geq 2$, we have that $|\rho(S)| \geq 2$. That implies that ρ is splitting.

In each case of the definition of $\rho(S)$ and for each $A \in \rho(S)$, we will determine a *positive* real number $k_{A,S}$ (in general, depending on α) such that $\Delta(S, S_A) = \mu(S) - \mu(S_A) \geq k_{A,S}$. We will use these values to state the equation (3), whose root τ'_S provides, by Lemma 8, an upper bound for τ_S, the root of the equation (1). The fact that the numbers $k_{A,S}$ are positive implies that the cover function ρ is μ-compatible.

Once we describe all the cases, we will then finally select α. We will do it so that in each case considered when defining $\rho(S)$, the values $k_{A,S}$ are positive and $\tau'_S \leq 1.6701..$. That property shows that ρ is μ-bounded by $1.6701..$.

The properties of the cover function ρ and of the function σ together with Lemma 7 imply now Lemma 12.

In Cases 1 - 3 of the definition of the function ρ we deal with three simple situations when $|At(S)| = 1$, when S contains a 1-clause, and when some atom appears only negated in clauses of S.

Cases 4 - 10 cover situations when S contains a 2-clause. Case 4 covers the situation when there is a pair of 2-clauses in S with a common atom. Therefore in the remaining cases, we assume that the sets of atoms of 2-clauses in S are pairwise disjoint. That assumption makes it easier to analyze $\mu(S) - \mu(S_A)$ and obtain bounds $k_{S,A}$. Indeed, under that assumption, $k(S)$ is simply equal to the number of 2-clauses in S.

Cases 5 and 6 together cover the situations when there is a 2-clause and a 3-clause in S such that the 2-clause has a literal, which is dual to a literal occurring in the 3-clause, and when the set of atoms of the 2-clause is a subset of the set of atoms of the 3-clause. That allows us to assume in all subsequent cases that for every atom a occurring in a 2-clause c in S, all occurrences of a in clauses of S are positive. Indeed, they cannot be all negative by Case 3. Moreover, by Cases 1 and 4, all clauses in S that contain a and are different from c are 3-clauses. If any of them contains a negated occurrence of a, Case 5 or 6 would apply.

In Case 7 we assume that there is an atom of a 2-clause that does not belong to any other clause in S. In Case 8 we consider the situation when some atom of a 2-clause appears as a literal in exactly one 3-clause. Finally, in Cases 9 and 10 we assume that there is an atom of a 2-clause, which belongs to at least 2 different 3-clauses.

It is easy to verify that Cases 1 - 10 exhaust all possibilities when there is a 2-clause or a 1-clause in S. Therefore, from now on we assume that all clauses in S are 3-clauses. For an atom $a \in At(S)$, we denote by $T(a)$ the theory consisting of the clauses of S, in which a is one of the literals.

In Case 11 we consider theories S such that, for some atom $a \in At(S)$, there are two clauses in $T(a)$ such that one of them contains a literal, which is dual to a literal in the other one. Thus, in the remaining cases we assume that, for each atom $a \in At(S)$, the theory $T(a)$ does not contain dual literals. We denote by $\Gamma(a)$ an undirected graph whose vertices are the literals different from a occurring in the clauses of $T(a)$ and a pair of literals $\beta\gamma$ is an edge in $\Gamma(a)$ if $a \vee \beta \vee \gamma$ is a clause in $T(a)$. We call the number of neighbors of a vertex in a graph $\Gamma(a)$ the *degree* of the vertex.

In Cases 12 - 20 we assume that there is an atom $a \in At(S)$, for which the graph $\Gamma(a)$ has some specified structural properties. In Case 12 we assume that for some atom a, there is a vertex in the graph $\Gamma(a)$ with at least 5 neighbors. Case 13 covers the situation when the maximum degree of a vertex in some graph $\Gamma(a)$ is 3 or 4. In Case 14 we assume that there is an atom a such that $\Gamma(a)$ has at least 4 independent edges. In Cases 15 - 20 we consider the theories S such that the graph $\Gamma(a)$, for some $a \in At(S)$, has no vertices of degree 3 or more and is not isomorphic to any of the following three graphs: $C_3 \cup P_1$ (a graph whose components are a triangle and a single edge), $P_3 \cup P_1$ (a graph whose components are a 3-edge path and a single edge) and $3K_2$ (a graph whose components are three single edges).

Finally, in Case 21 we assume that, for all atoms $a \in At(S)$, the graphs $\Gamma(a)$ are isomorphic to one of the graphs $C_3 \cup P_1$, $P_3 \cup P_1$, $3K_2$. First we consider the case when some atom of $At(S)$ occurs in S negated. Next we assume that all occurrences of atoms in S are positive.

Let us now explain the choice of a particular value of α in the definition of the measure $\mu(S) = |At(S)| - \alpha k(S)$. Our goal is to get as good an upper bound for the number of leaves in $\mathcal{T}_T$ as we can. We choose the value of α so that the maximum τ_0 of the solutions of the equation (3) over all cases considered in the definition of ρ be as small as possible.

It turns out that the Cases 9(iii) and 14 are, in a sense, "extremal". In Case 9(iii) of the definition of $\rho(S)$, the equation (3) specializes to

$$\tau^{-1+\alpha} + \tau^{-4+3\alpha} + 2\tau^{-6+4\alpha} + \tau^{-8+5\alpha} = 1. \tag{4}$$

In Case 14, the equation (3) becomes

$$\tau^{-1} + \tau^{-1-4\alpha} = 1. \tag{5}$$

It is easy to verify that the value $\tau_1 = \tau_1(\alpha)$ of the positive root of the equation (4) satisfies the inequality $\tau_1 > 1$ and grows, when α grows from 0 to 1. On the other hand, the value $\tau_2 = \tau_2(\alpha)$ of the positive root of the equation (5) satisfies the inequality $\tau_2 > 1$ and decreases, when α grows from 0 to 1. The larger of the roots τ_1, τ_2 is minimized when $\tau_1 = \tau_2$. This equality happens to be achieved for $\alpha = 0.1950..$. For this value of α, we have $\tau_1 = \tau_2 = 1.6701..$. Moreover, it can be checked by direct computations that in all remaining cases, if $\alpha = 0.1950..$, then the values $k_{S,A}$ are positive and the roots of the equation (3) are smaller than 1.67. Thus, for $\alpha = 0.1950..$, ρ is μ-compatible and μ-bounded by $1.6701..$.

8 The case of an arbitrary $t \geq 2$

In this section, we briefly discuss computing minimal models of t-CNF theories, stable models of normal t programs and answer sets of disjunctive t-programs for an arbitrary $t \geq 2$. First, we recall the following result from (Lonc and Truszczyński 2003). When stating it, by α_t we denote the unique positive root of the equation $1 + \tau + \tau^2 + \ldots + \tau^{t-1} = \tau^t$. It is easy to see that $\alpha_t \leq 2 - 1/2^t$ and one can show numerically that $\alpha_2 = 1.6180..$, $\alpha_3 = 1.8393..$, $\alpha_4 = 1.9275..$ and $\alpha_5 = 1.9659..$.

Theorem 6 (*(Lonc and Truszczyński 2003)*)
Let t be an integer, $t \geq 2$. There is an algorithm computing all stable models of t-programs that runs in time $O(m\alpha_t^n)$.

Theorem 6 can also be derived for the results we presented in this paper. Let T be a CNF theory that contains a t-clause $\beta_1 \vee \ldots \vee \beta_t$. It is easy to see that the family of sets $\{A_1, \ldots, A_t\}$, where $A_i = \{\overline{\beta}_1, \ldots, \overline{\beta}_{i-1}, \beta_i\}$, $1 \leq i \leq t$, is a cover for T. By exploiting that observation, we can show that there is a splitting cover function ρ such that for every t-CNF theory T,

1. $\rho(T)$ can be computed in time $O(m)$, and
2. $|L(\mathcal{T}_T)| \leq \alpha_t^n$ (we use the identity function for the simplifying function σ and we use the measure $\mu(S) = |At(S)|$ to derive that bound).

It follows that the corresponding implementation of the algorithm min^+ runs in $O(m\alpha_t^n)$ steps. Moreover, the family $\mathcal{M}^+(T, \emptyset)$ that is returned by $min^+(T, T, \emptyset)$ satisfies $|\mathcal{M}^+(T, \emptyset)| \leq \alpha_t^n$. Reasoning as in other places in the paper, it is easy to derive Theorem 6 from these observations.

Since $\mathcal{M}^+(T, \emptyset)$ contains all minimal models of T, we also get the following result.

Theorem 7
Every t-CNF theory T (every normal t-program P and every disjunctive t-program P, respectively) has at most $\alpha_t^n \leq (2 - 1/2^t)^n$ minimal models (stable models, answer-sets, respectively).

Next, we will construct algorithms for computing all minimal models of t-CNF theories and all answer sets of disjunctive t-programs in the case of an arbitrary $t \geq 2$. Not surprisingly, in the case of $t = 2$ and $t = 3$ these results are weaker than those we obtained earlier in the paper. Our approach does not depend on general results developed earlier in Sections 3 and 4. It exploits instead recent results on deciding satisfiability of t-CNF theories (Dantsin et al. 2002).

Theorem 8
There is an algorithm to compute all minimal models of t-CNF theories and answer sets of disjunctive t-programs, respectively, that runs in time $O(q(m)(3 - 2/(t + 1))^n)$, for some polynomial q.

Proof
There is a polynomial q' such that the satisfiability of t-CNF theories can be decided in time $O(q'(m)(2 - 2/(t + 1))^n)$ (Dantsin et al. 2002). Thus, by Proposition 2, there is an an algorithm to decide whether a set $M \subseteq At(T)$ is a minimal model of a t-CNF theory T, which runs in time

$$O(|M|q'(m + 1)(2 - 2/(t + 1))^{|M|}) = O(q(m)(2 - 2/(t + 1))^{|M|}),$$

where $q(m) = mq'(m + 1)$. Using this algorithm as a minimality-testing procedure in the straightforward algorithm to compute all minimal models of a t-CNF theory T, which generates all subsets of $At(T)$ and tests each of them for being a minimal

model of T, yields a method to compute all minimal models of a t-CNF theory that runs in time:

$$O(\sum_{i=0}^{n} \binom{n}{i} q(m)(2-2/(t+1))^i) = O(q(m)(3-2/(t+1))^n.$$

The argument in the case of answer sets is similar. □

We note that the method we used to derive Theorem 5 can be generalized to an arbitrary $t \geq 3$ (using the observations made after Theorem 6). However, the bound $|\mathcal{M}^+(T,\emptyset)| \leq \alpha_t^n$ is too week to yield algorithms faster than the algorithm described in the proof of Theorem 8. The only exception is the case of $t = 4$, where by generalizing the proof of Theorem 5 we can derive an algorithm constructing all minimal models of 4-CNF theories (answer sets of disjunctive 4-programs) in $O(m2.5994..^n)$ steps. Since the improvement over the bound of $O(q(m)2.6^n)$ implied by Theorem 8 is so small, we omit the details of the derivation of the $O(m2.5994..^n)$ bound.

9 Lower bounds

In this section, we derive lower bounds on the number of minimal models, stable models or answer sets a t-CNF theory, a t-program and a disjunctive t-program may have. We will also derive lower bounds on the running time of algorithms for computing *all* minimal models, stable models or answer sets of t-CNF theories and t-programs.

All examples we construct have a similar structure. Let X be a set of atoms and let t be an integer such that $2 \leq t \leq |X|$. By $E_{t,X}$ we denote a t-CNF theory consisting of all clauses of the form $a_1 \vee \ldots \vee a_t$, where atoms a_i belong to X and are pairwise distinct. By $E^d_{t,X}$, we mean a disjunctive t-program consisting of the same clauses, but treated as disjunctive-program clauses. Finally, by $E^p_{t,X}$ we mean a t-program consisting of all program clauses of the form $a_t \leftarrow \mathbf{not}(a_1), \ldots, \mathbf{not}(a_{t-1})$, where, as before, all atoms a_i belong to X and are pairwise distinct. The sizes of $E_{t,X}$ and $E^d_{t,X}$ are the same and are equal to $t\binom{|X|}{t}$. The size of $E^p_{t,X}$ is $t^2\binom{|X|}{t}$.

Minimal models of $E_{t,X}$, stable models of $E^p_{t,X}$ and answer sets of $E^d_{t,X}$ coincide. In fact, in each case, they are precisely $(|X|-t+1)$-element subsets of X. Thus, $E_{t,X}$, $E^d_{t,X}$ and $E^p_{t,X}$ have $\binom{|X|}{t-1}$ minimal models, stable models and answer sets, respectively.

Let k be a positive integer and let us consider $k(2t-1)$ *distinct* elements $x_{i,j}$, where $i = 1, \ldots, k$ and $j = 1, \ldots, 2t-1$. We define $X_i = \{x_{i,1}, \ldots, x_{i,2t-1}\}$ and set

$$F_{t,k} = \bigcup_{i=1}^{k} E_{t,X_i}, \quad F^p_{t,k} = \bigcup_{i=1}^{k} E^p_{t,X_i} \quad \text{and} \quad F^d_{t,k} = \bigcup_{i=1}^{k} E^d_{t,X_i}.$$

Clearly, $F_{t,k}$ is a t-CNF theory, $F^p_{t,k}$ is a normal t-program and $F^d_{t,k}$ is a disjunctive t-program. Moreover, since each of these theories (programs) is the *disjoint* union of k *isomorphic* components E_{t,X_i} (E^d_{t,X_i}, E^p_{t,X_i}, respectively), we have the following simple observations:

1. $|At(F_{t,k})| = |At(F^p_{t,k})| = |At(F^d_{t,k})| = k(2t-1)$.

2. The size of $F_{t,k}$ and $F^d_{t,k}$ is $kt\binom{2t-1}{t}$; the size of $F^p_{t,k}$ is $kt^2\binom{2t-1}{t}$.
3. $F_{t,k}$, $F^p_{t,k}$ and $F^d_{t,k}$ have $\binom{2t-1}{t}^k$ minimal models, stable models and answer sets, respectively, and each of these models or answer sets has kt elements.

Let us define $\mu_t = \binom{2t-1}{t}^{1/2t-1}$. The observations (1)-(3) imply the following result.

Theorem 9
Let t be an integer, $t \geq 2$. There are positive constants d_t, D_t and D'_t such that for every $n \geq 2t-1$ there is a t-CNF theory T (a t-program P or a disjunctive t-program Q, respectively) with n atoms and such that

1. The size m of T (P and Q, respectively) satisfies $m \leq d_t n$
2. The number of minimal models of T (stable models of P or answer sets of Q, respectively) is at least $D_t \mu_t^n$ and the sum of their cardinalities is at least $D'_t n \mu_t^n$.

Proof
We will prove the assertion only in the case of CNF theories. The arguments for t-programs and disjunctive t-programs are similar.

We will show that $d_t = 2\binom{2t-1}{t}$, $D_t = \mu_t^{-(2t-1)}$ and $D'_t = D_t/4$ have the required properties.

Let $n \geq 2t-1$. We select k to be the largest integer such that $k(2t-1) \leq n$. Clearly, $k \geq 1$. We select a set X of $n - k(2t-1)$ atoms, all of them different from atoms $x_{i,j}$ that appear in the theory $F_{t,k}$. Finally, we define $T = F_{t,k} \cup E_{t,X}$.

Clearly, T contains n atoms. Moreover, the size of T, m, satisfies $m = m' + m''$, where m' and m'' are the sizes of $F_{t,k}$ and $E_{t,X}$, respectively. Since $k \geq 1$, $m'' < m'$. Thus, $m \leq 2m' = 2kt\binom{2t-1}{t}$ (Observation 2). Since $kt \leq k(2t-1) \leq n$, we obtain $m \leq nd_t$.

The number of minimal models of T is greater than or equal to the number of minimal models of $F_{t,k}$ (since the sets of atoms of $F_{t,k}$ and $E_{t,X}$ are disjoint, every minimal model of $F_{t,k}$ extends to a minimal model of T). By Observation 3, the latter number is $\mu_t^{k(2t-1)}$. We now have

$$\mu_t^{k(2t-1)} \geq \mu_t^{n-(2t-1)} = \mu_t^{-(2t-1)}\mu_t^n = D_t\mu_t^n.$$

Each of the minimal models has size at least kt (again, by the fact that the sets of atoms of $F_{t,k}$ and $E_{t,X}$ are disjoint). Since $kt = \lfloor n/(2t-1) \rfloor t \geq n/4$, the total size of all minimal models of T is at least $D'_t n \mu_t^n$. □

As a corollary to Theorem 9, we obtain the following result.

Corollary 4
Let t be an integer, $t \geq 2$.

1. There is a t-CNF theory (a t-program, a disjunctive t-program) with n atoms and $\Omega(\mu_t^n)$ minimal models (stable models, answer sets, respectively).

2. Every algorithm computing all minimal models of t-CNF theories (stable models of t-programs, answer sets of disjunctive t-programs, respectively) requires in the worst case at least $\Omega(n\mu_t^n)$ steps.
3. Let $0 < \alpha < \mu_t$. For every polynomial f, there is no algorithm for computing all minimal models of t-CNF theories (stable models of t-programs, answer sets of disjunctive t-programs, respectively) with worst-case performance of $O(f(m)\alpha^n)$.

The lower bound given by Corollary 4(1) specializes to (approximately) $\Omega(1.4422..^n)$ and $\Omega(1.5848..^n)$, for $t = 2$ and 3, respectively. Similarly, the lower bound given by Corollary 4(2) specializes to (approximately) $\Omega(n1.4422..^n)$ and $\Omega(n1.5848..^n)$, for $t = 2$ and 3, respectively.

10 Discussion

The algorithms we presented in the case of 2-CNF theories, and normal and disjunctive 2-programs have worst-case performance of $O(m1.4422..^n)$. The algorithm we designed for the task of computing stable models of normal 3-programs runs in time $O(m1.6701..^n)$. Finally, our algorithms for computing minimal models of 3-CNF theories and answer sets of disjunctive logic programs run in time $O(mn^2 2.2782..^n)$. All these bounds improve by exponential factors over the corresponding straightforward ones.

The key question is whether still better algorithms are possible. In this context, we note that our algorithms developed for the case of 2-CNF theories and 2-programs are optimal, as long as we are interested in *all* minimal models, stable models and answer sets, respectively. However, we can compute a *single* minimal model of a 2-CNF theory T or decide that T is unsatisfiable in *polynomial* time. Indeed, it is well known that we can compute a model M of T or decide that T is unsatisfiable in polynomial time. In the latter case, no minimal models exist. In the former one, by the proof of Proposition 2, M is a minimal model of T if and only if theories $T_L \cup \{\overline{a}\}$, where $L = \{\overline{b}\colon b \in At(T) - M\}$ and $a \in M$, are all unsatisfiable. Thus, by means of polynomially many satisfiability checks we either determine that M is a minimal model of T or find $a \in M$ such that $M - \{a\}$ is a model of T. In contrast, deciding whether a 2-program has a stable model and whether a disjunctive 2-program has an answer set is NP-complete. Thus, it is unlikely that there are polynomial-time algorithms to compute a single stable model (answer set) of a (disjunctive) 2-program or decide that none exist. Whether our bound of $O(m1.4422..^n)$ can be improved by an exponential factor if we are interested in computing a single stable model or a single answer set, rather than all of them, is an open problem.

The worst-case behavior of our algorithms designed for the case of 3-CNF theories and 3-programs does not match the lower bound of $O(n1.5848..^n)$ implied by Corollary 4. Thus, there is still room for improvement, even when we want to compute *all* minimal models, stable models and answer sets. In fact, we conjecture that exponentially faster algorithms exist.

In the case of 3-CNF theories, reasoning similarly as in the case of 2-CNF theories, and using the proof of Proposition 2 and the algorithm from (Dantsin et al. 2002), shows that in time $O(p(m)1.481^n)$, where p is a polynomial, one can compute *one* minimal model of a 3-CNF theory T or determine that T is unsatisfiable. This is a significantly better bound than $O(mn^2 2.2782..^n)$ that we obtained for computing *all* minimal models. We do not know however, whether the bound $O(p(m)1.481^n)$ is optimal. Furthermore, we do not know whether an exponential improvement over the bound of $O(mn^2 2.2782..^n)$ is possible if we want to compute a single answer set of a disjunctive 3-program or determine that none exists. Similarly, we do not know whether one can compute a single stable model of a 3-program or determine that none exists in time exponentially lower than $O(m1.6701..^n)$.

In some cases, our bound in Theorem 5 can be improved. Let $\mathcal{F}$ be the class of all CNF theories consisting of clauses of the form $a_1 \vee \ldots \vee a_p$ or $a \vee \overline{b}$, where $a_1, \ldots, a_p$, a and b are atoms. Similarly, let $\mathcal{G}$ be the class of all disjunctive programs with clauses of the form $a_1 \vee \ldots \vee a_p \leftarrow \mathbf{not}(b_1), \ldots, \mathbf{not}(b_r)$ or $a \leftarrow b, \mathbf{not}(b_1), \ldots, \mathbf{not}(b_r)$, where $a_1, \ldots, a_p$, $b_1, \ldots, b_r$, a and b are atoms. Checking whether a set M is a minimal model of a theory from $\mathcal{F}$ or an answer set of a program from $\mathcal{G}$ is in the class P (the task can be accomplished in linear time by extending the argument we used to establish Proposition 1). Thus, by Lemma 9, we obtain the following result.

Theorem 10
There is an algorithm to compute minimal models of 3-CNF theories in $\mathcal{F}$ (answer sets of disjunctive 3-programs in $\mathcal{G}$, respectively), that runs in time $O(m1.6701..^n)$.

Finally, we stress that our intention in this work was to better understand the complexity of problems to compute stable models of programs, minimal models of CNF theories and answer sets of disjunctive programs. Whether our theoretical results and algorithmic techniques we developed here will have any significant practical implications is a question for future research.

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Appendix

We complete here the proof of Lemma 12, outlined in Section 7. We defined there a measure μ by setting

$$\mu(S) = n(S) - \alpha k(S),$$

where $n(S)$ is the number of atoms in S, $k(S)$ is the maximum number of 2-clauses in S with mutually disjoint sets of atoms, and $\alpha = 0.1950..$.

We introduced in Section 7 a simplifying function σ, which eliminates from a 3-CNF theory T repeated occurrences of the equivalent clauses and also all 3-clauses subsumed by 2-clauses in T. We showed there that σ is μ-compatible and that $\sigma(T)$ can be computed in linear time in the size of T.

To complete the proof, we still need to define a cover function ρ postulated in Lemma 12. As we argued in Section 7, it is enough to define $\rho(S)$ when S is a 3-CNF theory with at least one atom and such that S contains no multiple occurrences of clauses nor 3-clauses subsumed by a 2-clause in S. In addition, we will assume that no clause in S contains multiple occurrences of a literal or a pair of dual literals (given a linked-list representation of S, this condition can be enforced, if necessary, in linear time in the size of S).

For each such theory S we will define $\rho(S)$. To this end, we consider 21 cases that we specified in Section 7. When discussing a case, we assume that none of the previously considered cases applies.

In each case, we describe $\rho(S)$. Then, for each $A \in \rho(S)$, we find a positive real number $k_{A,S}$ such that $\Delta(S, S_A) = \mu(S) - \mu(S_A) \geq k_{A,S}$. To this end, we find lower bounds for $n(S) - n(S_A)$ and $k(S_A) - k(S)$. Clearly, $|At(A)| \leq n(S) - n(S_A)$ and we use $|At(A)|$ to estimate $n(S) - n(S_A)$ from below. In each case we consider, the cardinality of $At(A)$ is equal to the number of literals we list as members of A. In Cases 2 - 5, we provide explicit proofs of that claim. In all other cases arguments are similar and we only outline them or omit them altogether. As concerns $k(S_A) - k(S)$, we provide lower bounds and arguments to justify them in each case we consider. For the most part, the proofs take advantage of the fact that once we settle Case 4, we can assume that all 2-clauses in S have pairwise disjoint sets of atoms.

Establishing positive bounds $k_{A,S}$ for $\Delta(S, S_A)$ shows that ρ is μ-compatible. It also yields a specific instance of the equation (3). Assuming $\alpha = 0.1950..$, we find the root τ'_S of the equation (3). In each case, $\tau'_S \leq 1.6701..$. By Lemma 7, that implies that ρ is μ-bounded by $1.6701..$ and completes the proof of Lemma 12.

In what follows, we consistently use c, possibly with indices, to denote clauses. We use Greek alphabet letters to denote literals and Latin alphabet letters to denote atoms. Throughout the proof, we strictly adhere to the assumption that b, d, e, g, ℓ and o denote atoms appearing in literals β, δ, ϵ, γ, λ and ω, respectively, and we extend this notation to the case when we use these letters together with indices. Finally, we consistently view clauses as disjunctions of their literals.

We denote by I a largest set of 2-clauses in S with pairwise disjoint sets of atoms.

Case 1. $|At(S)| = 1$, say $At(S) = \{b\}$.

In this case, $S = \{\beta\}$ or $S = \{\beta, \overline{\beta}\}$. Clearly,

$$\mathcal{A} = \{\{\beta\}\}$$

is a cover for S and we set $\rho(S) = \mathcal{A}$. It is easy to see that $At(S_A) = 0$. Moreover, since neither S nor S_A contain 2-clauses, $\Delta(S, S_A) = 1$. The equation (3) becomes

$$\tau = 1$$

and its root, τ'_S satisfies $\tau'_S = 1$.

Case 2. There is a 1-clause in S.

Let ω be the literal of a 1-clause in S. Since $|At(S)| \geq 2$, there is an atom $y \in At(S)$ such that $o \neq y$. It is evident that

$$\mathcal{A} = \{\{\omega, y\}, \{\omega, \overline{y}\}\}$$

is a cover for S and we set $\rho(S) = \mathcal{A}$.

For every $A \in \mathcal{A}$, the theory S_A contains all 2-clauses of I that do not contain o or y. Thus $k(S_A) \geq k(S) - 2$. It follows that for $A \in \mathcal{A}$,

$$\Delta(S, S_A) \geq 2 - 2\alpha.$$

Moreover, since $\alpha < 1$, for every $A \in \mathcal{A}$, $\Delta(S, S_A) > 0$. The equation (3) becomes

$$2\tau^{-2+2\alpha} = 1.$$

and $\tau'_S \leq 1.54$.

Case 3. There is an atom a that appears negated in every clause in S.

Clearly, a does not belong to any minimal model of S or, in other words, every *minimal* model of S is consistent with $\{\overline{a}\}$. Let y be an atom in $At(S)$ such that $y \neq a$. It follows that

$$\mathcal{A} = \{\{\overline{a}, y\}, \{\overline{a}, \overline{y}\}\}$$

is a cover for S and we set $\rho(S) = \mathcal{A}$. Reasoning as Case 2, we obtain that for every $A \in \mathcal{A}$,

$$\Delta(S, S_A) \geq 2 - 2\alpha > 0.$$

Moreover, the equation (3) becomes

$$2\tau^{-2+2\alpha} = 1$$

and $\tau'_S \leq 1.54$.

Case 4. There are two 2-clauses in S, which contain a common atom.

Because of the assumption we adopt for Case 4, there is a clause, say c_1 such that $c_1 \notin I$. Since I is a largest set of 2-clauses in S with pairwise disjoint sets of atoms, there is a clause, say c_2, in I such that c_1 and c_2 have a common atom.

Subcase (i). $c_1 = \overline{\omega} \vee \gamma$ and $c_2 = \omega \vee \beta$ (that is, the common atom appears in c_1 and c_2 "in the opposite ways").

We note that every model consistent with ω is also consistent with γ, as it satisfies the clause c_1. Similarly, every model consistent with $\overline{\omega}$ is also consistent with β. Thus,

$$\mathcal{A} = \{\{\omega, \gamma\}, \{\overline{\omega}, \beta\}\}.$$

is a cover for S and we define $\rho(S) = \mathcal{A}$.

Clearly, $|At(\{\omega, \gamma\})| = 2$ (otherwise, the 2-clause c_1 would contain a multiple occurrence of an atom or a pair of dual literals). Similarly, $|At(\{\overline{\omega}, \beta\})| = 2$, as well.

The theory $S_{\{\omega,\gamma\}}$ contains all 2-clauses of I except for c_2 and, possibly, a clause in I containing the atom of γ. Thus, $k(S_{\{\omega,\gamma\}}) \geq k(S)-2$. The theory $S_{\{\overline{\omega},\beta\}}$, on the other hand, contains all 2-clauses of I except for c_2. Hence, $k(S_{\{\overline{\omega},\beta\}}) \geq k(S) - 1$. It follows that

$$\Delta(S, S_A) \geq \begin{cases} 2 - 2\alpha & \text{if } A = \{\omega, \gamma\} \\ 2 - \alpha & \text{if } A = \{\overline{\omega}, \beta\}. \end{cases}$$

Since $\alpha < 1$, $2 - 2\alpha > 0$ and $2 - \alpha > 0$. Thus, for every $A \in \mathcal{A}$, $\Delta(S, S_A) > 0$. Moreover, with $2-2\alpha$ and $2-\alpha$ as numbers $k_{A,S}$, $A \in \mathcal{A}$, the equation (3) becomes

$$\tau^{-2+2\alpha} + \tau^{-2+\alpha} = 1.$$

For $\alpha = 0.1950..$ its root, τ'_S, satisfies $\tau'_S \leq 1.51$.

Comment. From now on we will not explicitly state the numbers $k_{A,S}$. We will specify them implicitly in inequalities bounding $\Delta(S, S_A)$ from below. In each case, it will be straightforward to see that the numbers are positive, due to the fact that $\alpha < 1$. In each case, we will present an instance of the equation (3), implied by the bounds established in the case, as well as the root of the equation, computed under the assumption that $\alpha = 0.1950..$.

Subcase (ii). $c_1 = \omega \vee \beta$, $c_2 = \omega \vee \gamma$ (that is, the common atom to c_1 and c_2 appears in c_1 and c_2 "in the same way").

Every model consistent with $\overline{\omega}$ is consistent with β and γ. Thus, the following family

$$\mathcal{A} = \{\{\omega\}, \{\overline{\omega}, \beta, \gamma\}\}$$

is a cover for S. We use it as the value of $\rho(S)$.

Since c_1 and c_2 are 2-clauses of S, $o \neq b, g$. Moreover, as $c_1 \notin I$ and $c_2 \in I$, $c_1 \neq c_2$ and, consequently, $\beta \neq \gamma$. We can also assume that $\beta \neq \overline{\gamma}$ (if that was not the case, Subcase (i) would apply). Thus, $b \neq g$ and $|At(\{\overline{\omega}, \beta, \gamma\})| = 3$.

The theory $S_{\{\omega\}}$ contains all 2-clauses of I except for c_2. Thus $k(S_{\{\omega\}}) \geq k(S)-1$. The theory $S_{\{\overline{\omega},\beta,\gamma\}}$ contains all 2-clauses of $I - \{c_2\}$ that do not contain g (the atom of γ). Thus $k(S_{\{\overline{\omega},\beta,\gamma\}}) \geq k(S) - 2$. It follows that for every $A \in \mathcal{A}$, we have

$$\Delta(S, S_A) \geq \begin{cases} 1 - \alpha & \text{if } A = \{\omega\} \\ 3 - 2\alpha & \text{if } A = \{\overline{\omega}, \beta, \gamma\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1+\alpha} + \tau^{-3+2\alpha} = 1$$

and $\tau'_S \leq 1.58..$ (we recall that we take $\alpha = 0.1950..$).

Comment. There are no other possibilities in Case 4. From now on we will assume that the set of *all* 2-clauses in S consists of clauses which do not have common atoms. Thus, $k(S)$ is simply equal to the number of *all* 2-clauses in S. Moreover,

when simplifying S with respect to a literal ω (removing clauses subsumed by ω and eliminating $\overline{\omega}$ from other clauses of S as part of the computation of S_A), at most one 2-clause will be eliminated in the process and $k(S)$ will decrease at most by 1. Throughout the proof, we denote that set of 2-clauses of S by I.

Case 5. There are clauses $c_1 = \omega \vee \beta \vee \gamma$ and $c_2 = \overline{\beta} \vee \delta$ in S such that $d \notin At(c_1)$.

Subcase (i). Neither o nor g is an atom of a 2-clause in S.

The family

$$\mathcal{A} = \{\{\beta, \delta\}, \{\overline{\beta}\}\}$$

is a cover for S (indeed, if a model of S is not consistent with $\overline{\beta}$, it is consistent with β and, due to clause c_2, with δ). We define $\rho(S) = \mathcal{A}$.

Since $\overline{\beta} \vee \delta$ is a clause in S, $b \neq d$ and, consequently, $|At(\{\beta, \delta\})| = 2$. Moreover, all 2-clauses in $I - \{c_2\}$ are 2-clauses of $S_{\{\beta,\delta\}}$ and $S_{\{\overline{\beta}\}}$. Thus, $k(S_{\{\beta,\delta\}}) \geq k(S) - 1$. We also have that $c_3 = \omega \vee \gamma$ is a 2-clause in $S_{\{\overline{\beta}\}}$. Since S contains no 3-clauses subsumed by 2-clauses in S, $c_3 \notin S$. By the assumption adopted for this subcase, no atom of c_3 belongs to any 2-clause in $I - \{c_2\}$. Thus, $k(S_{\{\overline{\beta}\}}) \geq k(S)$ and, for every $A \in \mathcal{A}$,

$$\Delta(S, S_A) \geq \begin{cases} 2 - \alpha & \text{if } A = \{\beta, \delta\} \\ 1 & \text{if } A = \{\overline{\beta}\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-2+\alpha} + \tau = 1$$

and $\tau'_S \leq 1.67$.

Subcase (ii). There is a 2-clause $c_3 \in I$ of the form $c_3 = \overline{\omega} \vee \epsilon$ or $c_3 = \overline{\gamma} \vee \epsilon$. We will assume $c_3 = \overline{\omega} \vee \epsilon$. The other case is symmetric.

Every model of S consistent with β is consistent with δ (clause c_2). Every model consistent with $\{\overline{\beta}, \omega\}$ is consistent with ϵ (clause c_3). Finally, every model consistent with $\{\overline{\beta}, \overline{\omega}\}$ is consistent with γ (clause c_1). Therefore, the family

$$\mathcal{A} = \{\{\beta, \delta\}, \{\overline{\beta}, \omega, \epsilon\}, \{\overline{\beta}, \overline{\omega}, \gamma\}\}$$

is a cover for S. We define $\rho(S) = \mathcal{A}$.

Since $\overline{\beta} \vee \delta$ is a clause in S, $b \neq d$. Thus, $|At(\{\beta, \delta\})| = 2$. Similarly, since c_1 is a 3-clause in S, the atoms b, o and g are pairwise distinct and $At(\{\overline{\beta}, \overline{\omega}, \gamma\})| = 3$. Since $d \notin At(c_1)$, $d \neq o$. We already noted that $b \neq o$. Thus, $c_2 \neq c_3$. Consequently, $At(c_2) \cap At(c_3) = \emptyset$ (by the assumption made after Case 4) and $e \neq b$. Finally, $o \neq e$, as o and e appear together in c_3. Thus, $|At(\{\overline{\beta}, \omega, \epsilon\})| = 3$.

All 2-clauses of S except for c_2 are 2-clauses in $S_{\{\beta,\delta\}}$ and it follows that $k(S_{\{\beta,\delta\}}) \geq k(S) - 1$. All 2-clauses of S except for c_2 and c_3 are 2-clauses in $S_{\{\overline{\beta},\omega,\epsilon\}}$. Hence, $k(S_{\{\overline{\beta},\omega,\epsilon\}}) \geq k(S) - 2$. Finally, all 2-clauses in $I - \{c_2, c_3\}$ that do not contain g are 2-clauses in $S_{\{\overline{\beta},\overline{\omega},\gamma\}}$ and, consequently, $k(S_{\{\overline{\beta},\overline{\omega},\gamma\}}) \geq k(S) - 3$. Hence, for every $A \in \mathcal{A}$,

$$\Delta(S, S_A) \geq \begin{cases} 2 - \alpha & \text{if } A = \{\beta, \delta\} \\ 3 - 2\alpha & \text{if } A = \{\overline{\beta}, \omega, \epsilon\} \\ 3 - 3\alpha & \text{if } A = \{\overline{\beta}, \overline{\omega}, \gamma\}. \end{cases}$$

These bounds yield the following instance of the equation (3):

$$\tau^{-2+\alpha} + \tau^{-3+2\alpha} + \tau^{-3+3\alpha} = 1.$$

Its root τ'_S satisfies $\tau'_S \leq 1.64$.

Subcase (iii). There is a 2-clause $c_3 \in I$ of the form $c_3 = \omega \vee \epsilon$ and g does not belong to any 2-clause in I (or the symmetric situation with the roles of ω and γ interchanged).

Every model of S consistent with β is consistent with δ (due to clause c_2). Moreover, every model consistent with $\{\overline{\beta}, \overline{\omega}\}$ is consistent with $\{\gamma, \epsilon\}$ (clauses c_1 and c_3). Therefore, the family

$$\mathcal{A} = \{\{\beta, \delta\}, \{\overline{\beta}, \omega\}, \{\overline{\beta}, \overline{\omega}, \gamma, \epsilon\}\}$$

is a cover for S and we take it as the value of $\rho(S)$.

Arguing as in the previous subcase, we show that $e \neq o, b$. Moreover, we observe that $e \neq g$, as we assumed that g does not belong to any 2-clause in I. Thus, the cardinality of the last set in $\mathcal{A}$ is 4 (it is easy to see that each of the other two sets has 2 elements).

All 2-clauses of S except for c_2 are still 2-clauses in $S_{\{\beta,\delta\}}$. Thus, $k(S_{\{\beta,\delta\}}) \geq k(S) - 1$. Since the atom of γ does not belong to any 2-clause in I, all 2-clauses of S except for c_2 and c_3 are 2-clauses in $S_{\{\overline{\beta},\omega\}}$ and in $S_{\{\overline{\beta},\overline{\omega},\gamma,\epsilon\}}$. Thus, $k(S_{\{\overline{\beta},\omega\}}) \geq k(S) - 2$ and $k(S_{\{\overline{\beta},\overline{\omega},\gamma,\epsilon\}}) \geq k(S) - 2$. Hence, for every $A \in \mathcal{A}$,

$$\Delta(S, S_A) \geq \begin{cases} 2-\alpha & \text{if } A = \{\beta, \delta\} \\ 2-2\alpha & \text{if } A = \{\overline{\beta}, \omega\} \\ 4-2\alpha & \text{if } A = \{\overline{\beta}, \overline{\omega}, \gamma, \epsilon\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-2+\alpha} + \tau^{-2+2\alpha} + \tau^{-4+2\alpha} = 1,$$

and $\tau'_S \leq 1.67$.

Subcase (iv). There are 2-clauses $c_3, c_4 \in I$ of the form $c_3 = \gamma \vee \epsilon$ and $c_4 = \omega \vee \lambda$.

Every model consistent with $\{\overline{\omega}, \gamma\}$ is consistent with λ (clause c_4). Moreover, every model consistent with $\{\overline{\omega}, \overline{\gamma}\}$ is consistent with $\{\lambda, \epsilon, \beta, \delta\}$ (clauses c_4, c_3, c_1 and c_2). Therefore the family

$$\mathcal{A} = \{\{\omega\}, \{\overline{\omega}, \gamma, \lambda\}, \{\overline{\omega}, \overline{\gamma}, \lambda, \epsilon, \beta, \delta\}\}$$

is a cover for S and we choose it to define $\rho(S)$.

Since $d \notin At(c_1)$, $d \neq o, g$. Since b, o and g are the atoms of the 3-clause c_1, $b \neq o, g$. Thus, $c_2 \neq c_3$ and $c_2 \neq c_4$. Let us assume that $c_3 = c_4$. Since $\gamma \neq \omega$, it follows that $\epsilon = \omega$. Consequently, c_3 subsumes c_1, a contradiction. Thus, $c_3 \neq c_4$ and so, c_2, c_3 and c_4 are pairwise different. By Case 4, it follows that c_2, c_3 and c_4 have pairwise disjoint sets of atoms. Consequently, $|At(\{\overline{\omega}, \gamma, \lambda\})| = 3$ and $|At(\{\overline{\omega}, \overline{\gamma}, \lambda, \epsilon, \beta, \delta\})| = 6$.

All 2-clauses of S except for c_4 are 2-clauses in $S_{\{\omega\}}$. Thus, $k(S_{\{\omega\}}) \geq k(S) - 1$. All 2-clauses of S except for c_3 and c_4 are 2-clauses in $S_{\{\overline{\omega},\gamma,\lambda\}}$. Consequently,

$k(S_{\{\overline{\omega},\gamma,\lambda\}}) \geq k(S) - 2$. Finally, all 2-clauses of S except for c_2, c_3 and c_4 are 2-clauses in $S_{\{\overline{\omega},\overline{\gamma},\lambda,\epsilon,\beta,\delta\}}$ and so $k(S_{\{\overline{\omega},\overline{\gamma},\lambda,\epsilon,\beta,\delta\}}) \geq k(S) - 3$. Hence, for every $A \in \mathcal{A}$,

$$\Delta(S, S_A) \geq \begin{cases} 1-\alpha & \text{if } A = \{\omega\} \\ 3-2\alpha & \text{if } A = \{\overline{\omega}, \gamma, \lambda\} \\ 6-3\alpha & \text{if } A = \{\overline{\omega}, \overline{\gamma}, \lambda, \epsilon, \beta, \delta\}. \end{cases}$$

For these bounds, the equation (3) becomes

$$\tau^{-1+\alpha} + \tau^{-3+2\alpha} + \tau^{-6+3\alpha} = 1,$$

and $\tau'_S \leq 1.66$ (assuming $\alpha = 0.1950..$).

Case 6. There are clauses $c_1, c_2 \in S$ such that c_1 is a 2-clause, c_2 is a 3-clause and $At(c_1) \subseteq At(c_2)$.

Since c_1 does not subsume c_2, we can assume that $c_1 = \omega \vee \gamma$ and $c_2 = \omega' \vee \overline{\gamma} \vee \beta$, where $\omega' = \omega$ or $\omega' = \overline{\omega}$.

Subcase (i). The atom b does not occur in any 2-clause.

Every model consistent with $\overline{\gamma}$ is consistent with ω (clause c_1). Therefore the family

$$\mathcal{A} = \{\{\overline{\gamma}, \omega\}, \{\gamma\}\}$$

is a cover for S and we set $\rho(S) = \mathcal{A}$.

All 2-clauses of S except for c_1 are 2-clauses in the theories $S_{\{\overline{\gamma},\omega\}}$ and $S_{\{\gamma\}}$. It follows that $k(S_{\{\overline{\gamma},\omega\}}) \geq k(S) - 1$. Moreover, $c_3 = \omega' \vee \beta$ is a 2-clause in $S_{\{\gamma\}}$. Since o appears in a 2-clause c_1, o does not appear in any other 2-clause in S. Thus, c_3 is different from all 2-clauses in $S_{\{\gamma\}}$ that belong to $I - \{c_1\}$. By the assumption we adopted in this subcase, b does not belong to any 2-clause in $I - \{c_1\}$ either. Thus, $k(S_{\{\gamma\}}) \geq k(S)$. It follows that for every $A \in \mathcal{A}$,

$$\Delta(S, S_A) \geq \begin{cases} 2-\alpha & \text{if } A = \{\overline{\gamma}, \omega\} \\ 1 & \text{if } A = \{\gamma\}. \end{cases}$$

In this case, we obtain the following instance of (3):

$$\tau^{-2+\alpha} + \tau = 1.$$

Its root τ'_S satisfies $\tau'_S \leq 1.67$.

Subcase (ii). The atom b belongs to some 2-clause in S.

Let c_3 be a 2-clause in S that contains b. Then $c_3 = \overline{\beta} \vee \epsilon$ or $c_3 = \beta \vee \epsilon$, for some literal ϵ, where $e \neq b$. We note that $e \neq o, g$ (as all 2-clauses in S have pairwise disjoint sets of atoms). Thus, if $c_3 = \overline{\beta} \vee \epsilon$, Case 5 would apply to c_3 and c_2, a contradiction. It follows that $c_3 = \beta \vee \epsilon$.

Every model consistent with $\{\overline{\beta}, \overline{\gamma}\}$ is consistent with $\{\epsilon, \omega\}$ (clauses c_3 and c_1), and every model consistent with $\{\overline{\beta}, \gamma\}$ is consistent with $\{\epsilon, \omega'\}$ (clauses c_3 and c_2). Therefore the family

$$\mathcal{A} = \{\{\beta\}, \{\overline{\beta}, \overline{\gamma}, \epsilon, \omega\}, \{\overline{\beta}, \gamma, \epsilon, \omega'\}\}$$

is a cover for S and we use it as the value of $\rho(S)$ in this case.

Every 2-clause of $I - \{c_3\}$ is a 2-clause of $S_{\{\beta\}}$. Thus, $k(S_{\{\beta\}}) \geq k(S) - 1$. Moreover, every 2-clause of $I - \{c_1, c_3\}$ is a 2-clause in both $S_{\{\overline{\beta},\overline{\gamma},\epsilon,\omega\}}$ and $S_{\{\overline{\beta},\gamma,\epsilon,\omega'\}}$. Hence, $k(S_{\{\overline{\beta},\overline{\gamma},\epsilon,\omega\}}) \geq k(S) - 2$ and $k(S_{\{\overline{\beta},\gamma,\epsilon,\omega'\}}) \geq k(S) - 2$. It follows that for every $A \in \mathcal{A}$,

$$\Delta(S, S_A) \geq \begin{cases} 1 - \alpha & \text{if } A = \{\beta\} \\ 4 - 2\alpha & \text{if } A = \{\overline{\beta}, \overline{\gamma}, \epsilon, \omega\}, \{\overline{\beta}, \gamma, \epsilon, \omega'\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1+\alpha} + 2\tau^{-4+2\alpha} = 1.$$

Its root is $\tau'_S \leq 1.65$.

Comment. Let $\beta \vee \omega$ be a 2-clause in S. Let c be a clause in S such that $b \in At(c)$. If c is a 1-clause, Case 2 applies. If c is a 2-clause, Case 4 applies. Thus, we can assume that c is a 3-clause. If $\overline{\beta}$ is a literal of c and $o \notin At(c)$, then Case 5 applies. If $o \in At(c)$, then Case 6 applies. Thus, we can assume that every clause $c \in S$ such that $b \in At(c)$ contains β as its literal. If $\beta = \overline{b}$, Case 3 applies. Thus, from now on we can assume that atoms of 2-clauses of S have only positive occurrences in S.

Case 7. There is an atom a that occurs in a 2-clause of S, say $c_1 = a \vee b$, and in no other clause of S.

Every *minimal* model consistent with a is consistent with $\overline{b}$. Indeed, if a model M of S contains both a and b then $M - \{a\}$ is a model of S, too, as c_1 is the only clause in S that contains a. Moreover, every model consistent with $\overline{a}$ is consistent with b (clause c_1). Hence, the family

$$\mathcal{A} = \{\{a, \overline{b}\}, \{\overline{a}, b\}\}$$

is a cover for S and we define $\rho(S) = \mathcal{A}$.

Clearly, 2-clauses in $I - \{c_1\}$ are 2-clauses in both $S_{\{a,\overline{b}\}}$ and $S_{\{\overline{a},b\}}$. Thus, $k(S_{\{a,\overline{b}\}}) \geq k(S) - 1$ and $k(S_{\{\overline{a},b\}}) \geq k(S) - 1$. Hence, for every $A \in \mathcal{A}$,

$$\Delta(S, S_A) \geq 2 - \alpha > 0.$$

The equation (3) becomes:

$$2\tau^{-2+\alpha} = 1$$

and we have $\tau'_S \leq 1.47$.

Case 8. There is an atom a that occurs in a 2-clause of S, say $c_1 = a \vee b$, and there is exactly one other clause in S, say c_2, such that $a \in At(c_2)$.

Since 2-clauses in S do not have atoms in common, c_2 is a 3-clause. We will assume that $c_2 = a \vee \gamma \vee \delta$.

Every model consistent with $\overline{a}$ is consistent with b (clause c_1). Every *minimal* model consistent with $\{a, b\}$ is consistent with $\{\overline{\gamma}, \overline{\delta}\}$. Indeed, if a model M is consistent with $\{a, b, \gamma\}$ or $\{a, b, \delta\}$ then $M - \{a\}$ is a model of S, too, (as a belongs to two clauses c_1 and c_2 only). Hence, the family

$$\mathcal{A} = \{\{\overline{a}, b\}, \{a, \overline{b}\}, \{a, b, \gamma, \delta\}\}$$

is a cover for S and we take it as $\rho(S)$ in this case.

All 2-clauses in $I - \{c_1\}$ are 2-clauses in both $S_{\{\overline{a},b\}}$ and $S_{\{a,\overline{b}\}}$, which implies $k(S_{\{\overline{a},b\}}) \geq k(S) - 1$ and $k(S_{\{a,\overline{b}\}}) \geq k(S) - 1$. Moreover, all 2-clauses in I except for c_1 and 2-clauses in I that contain g and d (there are at most two such 2-clauses) are 2-clauses in $S_{\{a,b,\gamma,\delta\}}$. Thus, $k(S_{\{a,b,\gamma,\delta\}}) \geq k(S) - 3$. Consequently,

$$\Delta(S, S_A) \geq \begin{cases} 2 - \alpha & \text{if } A = \{\overline{a}, b\}, \{a, \overline{b}\} \\ 4 - 3\alpha & \text{if } A = \{a, b, \gamma, \delta\}. \end{cases}$$

The equation (3) becomes

$$2\tau^{-2+\alpha} + \tau^{-4+3\alpha} = 1$$

and $\tau'_S \leq 1.65$.

Case 9. For some atom a that occurs in a 2-clause, say c_1, there are two other clauses c_2 and c_3 such that $At(c_2) \cap At(c_3) = \{a\}$.

Since 2-clauses in S do not have atoms in common, c_2 and c_3 are 3-clauses. Throughout Case 9, we assume that $c_1 = a \vee b$, $c_2 = a \vee \gamma \vee \delta$ and $c_3 = a \vee \epsilon \vee \lambda$.

We have $b \notin At(c_i)$, $i = 2, 3$, as otherwise Case 6 would apply. Thus, since $At(c_2) \cap At(c_3) = \{a\}$, it follows that all atoms a, b, g, d, e and ℓ are pairwise different.

Subcase (i). The atoms g, d, e and ℓ do not belong to any 2-clause in S.

Clearly, the family

$$\mathcal{A} = \{\{a\}, \{\overline{a}, b\}\}$$

is a cover for S and we set $\rho(S) = \mathcal{A}$.

All 2-clauses in $I - \{c_1\}$ are 2-clauses in $S_{\{a\}}$ and $S_{\{\overline{a},b\}}$. It follows that $k(S_{\{a\}}) \geq k(S) - 1$. Moreover, the clauses $c_4 = \gamma \vee \delta$ and $c_5 = \epsilon \vee \lambda$ are 2-clauses in $S_{\{\overline{a},b\}}$. Since no 3-clause in S is subsumed by a 2-clause in S, $c_4, c_5 \notin S$. By the assumption we adopted for the current subcase, the atoms of c_4 and c_5 do not appear in any 2-clause of $I - \{c_1\}$. Thus, $k(S_{\{\overline{a},b\}}) \geq k(S) + 1$. It follows that for every $A \in \mathcal{A}$,

$$\Delta(S, S_A) \geq \begin{cases} 1 - \alpha & \text{if } A = \{a\} \\ 2 + \alpha & \text{if } A = \{\overline{a}, b\}. \end{cases}$$

The corresponding instance of the equation (3) is

$$\tau^{-1+\alpha} + \tau^{-2-\alpha} = 1$$

and $\tau'_S \leq 1.66$, for $\alpha = 0.1950..$.

Subcase (ii). The atoms g and d or the atoms e and ℓ do not belong to 2-clauses in S.

We will assume that e and ℓ do not belong to 2-clauses in S (the other case is symmetric). Furthermore, we can assume that g or d, say g, is an atom of a 2-clause (otherwise, Case 9(i) would apply). Let $c_4 = g \vee h$ be a 2-clause in S containing g (we note that the other literal in the clause must be an atom). We can assume that $d \neq h$ as, otherwise, $At(c_4) \subseteq At(c_2)$ and Case 6 would apply. Finally, we note that $c_2 = a \vee g \vee \delta$ (since there is a 2-clause in S containing g, g appears positively in every clause in S).

Every model of S consistent with $\overline{a}$ is consistent with b (clause c_1) and every model consistent with $\{\overline{a}, \overline{g}\}$ is consistent with $\{b, h, \delta\}$ (clauses c_1, c_4 and c_2). Hence, the family

$$\mathcal{A} = \{\{a\}, \{\overline{a}, g, b\}, \{\overline{a}, \overline{g}, b, h, \delta\}\}$$

is a cover for S and we set $\rho(S) = \mathcal{A}$.

All 2-clauses of $I - \{c_1\}$ are 2-clauses in $S_{\{a\}}$ and so, $k(S_{\{a\}}) \geq k(S) - 1$. Similarly, all 2-clauses of $I - \{c_1, c_4\}$ are 2-clauses of $S_{\{\overline{a}, g, b\}}$. In addition, $c_5 = \epsilon \vee \lambda$ is a 2-clause in $S_{\{\overline{a}, g, b\}}$. Reasoning as before we see that c_5 is not a 2-clause of S. Moreover, by the assumption we adopted earlier in this subcase, its atoms do not occur in 2-clauses of $I - \{c_1, c_4\}$. Thus, $k(S_{\{\overline{a}, g, b\}}) \geq k(S) - 1$. Finally, every clause in $I - \{c_1, c_4\}$ that does not contain d is a 2-clause of $S_{\{\overline{a}, \overline{g}, b, h, \delta\}}$. In addition, the clause $c_5 = \epsilon \vee \lambda$ is a 2-clause in $S_{\{\overline{a}, \overline{g}, b, h, \delta\}}$. As before, c_5 is not a 2-clause of S and it does not have atoms in common with other 2-clauses of $S_{\{\overline{a}, \overline{g}, b, h, \delta\}}$. Thus, $k(S_{\{\overline{a}, \overline{g}, b, h, \delta\}}) \geq k(S) - 2$. Hence,

$$\Delta(S, S_A) \geq \begin{cases} 1 - \alpha & \text{if } A = \{a\} \\ 3 - \alpha & \text{if } A = \{\overline{a}, g, b\} \\ 5 - 2\alpha & \text{if } A = \{\overline{a}, \overline{g}, b, h, \delta\}. \end{cases}$$

The bounds listed above are positive. The equation (3) becomes

$$\tau^{-1+\alpha} + \tau^{-3+\alpha} + \tau^{-5+2\alpha} = 1$$

and for $\alpha = 0.1950..$, $\tau'_S \leq 1.67$.

Subcase (iii). At least one of the atoms g and d belongs to a 2-clause, say c_4, such that $At(c_4) \cap At(c_3) = \emptyset$, and at least one of the atoms e and ℓ belongs to a 2-clause, say c_5, such that $At(c_5) \cap At(c_2) = \emptyset$.

Without loss of generality, we will assume that g and e have the postulated property, and that $c_4 = g \vee h$ and $c_5 = e \vee f$, for some atoms h and f. We can assume that $d \neq h$ and $\ell \neq f$ (otherwise, Case 6 would apply). Since $h \notin At(c_3)$ and $f \notin At(c_2)$, the atoms a, b, g, h, e, f, d and ℓ are pairwise different.

Let M be a model of S. If M is consistent with $\{\overline{a}, g, e\}$, then M is consistent with b (clause c_1). If M is consistent with $\{\overline{a}, g, \overline{e}\}$ then M is consistent with $\{b, f, \lambda\}$ (clauses c_1, c_5 and c_3). If M is consistent with $\{\overline{a}, \overline{g}, e\}$ then it is consistent with $\{b, h, \delta\}$ (clauses c_1, c_4 and c_2). Finally, if M is consistent with $\{\overline{a}, \overline{g}, \overline{e}\}$, then M is consistent with $\{b, h, f, \delta, \lambda\}$ (clauses c_1, c_4, c_5, c_2 and c_3). Hence, the family

$$\mathcal{A} = \{\{a\}, \{\overline{a}, g, e, b\}, \{\overline{a}, g, \overline{e}, b, f, \lambda\}, \{\overline{a}, \overline{g}, e, b, h, \delta\}, \{\overline{a}, \overline{g}, \overline{e}, b, h, f, \delta, \lambda\}\}$$

is a cover for S and we define $\rho(S) = \mathcal{A}$.

All 2-clauses in $I - \{c_1\}$ are 2-clauses in $S_{\{a\}}$ and so, $k(S_{\{a\}}) \geq k(S) - 1$. All 2-clauses in $I - \{c_1, c_4, c_5\}$ are 2-clauses in $S_{\{\overline{a}, g, e, b\}}$ and so, $k(S_{\{\overline{a}, g, e, b\}}) \geq k(S) - 3$. Similarly, 2-clauses in $I - \{c_1, c_4, c_5\}$ that do not contain ℓ (that condition excludes at most one such 2-clause) are 2-clauses in $S_{\{\overline{a}, g, \overline{e}, b, f, \lambda\}}$. Thus, $k(S_{\{\overline{a}, g, \overline{e}, b, f, \lambda\}}) \geq k(S) - 4$. All 2-clauses of $I - \{c_1, c_4, c_5\}$ that do not contain d are 2-clauses in $S_{\{\overline{a}, \overline{g}, e, b, h, \delta\}}$. Thus, $k(S_{\{\overline{a}, \overline{g}, e, b, h, \delta\}}) \geq k(S) - 4$. Finally, all 2-clauses of $I - \{c_1, c_4, c_5\}$ that do not contain d and ℓ (that condition excludes at most two

2-clauses) are 2-clauses in $S_{\{\overline{a},\overline{g},\overline{e},b,h,f,\delta,\lambda\}}$. Therefore, $k(S_{\{\overline{a},\overline{g},\overline{e},b,h,f,\delta,\lambda\}}) \geq k(S)-5$. Hence,

$$\Delta(S, S_A) \geq \begin{cases} 1-\alpha & \text{if } A = \{a\} \\ 4-3\alpha & \text{if } A = \{\overline{a}, g, e, b\} \\ 6-4\alpha & \text{if } A = \{\overline{a}, g, \overline{e}, b, f, \lambda\}, \{\overline{a}, \overline{g}, e, b, h, \delta\} \\ 8-5\alpha & \text{if } A = \{\overline{a}, \overline{g}, \overline{e}, b, h, f, \delta, \lambda\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1+\alpha} + \tau^{-4+3\alpha} + 2\tau^{-6+4\alpha} + \tau^{-8+5\alpha} = 1,$$

and $\tau'_S = 1.6701..$.

Comment. We can assume now that there is a 2-clause, say c_4, in S such that $|At(c_4) \cap \{g, d\}| = 1$ and $|At(c_4) \cap \{e, \ell\}| = 1$. Otherwise, one of the subcases (i)-(iii) would apply. Without loss of generality, we will assume that $c_4 = g \vee e$.

Subcase (iv). The atoms d and ℓ do not belong to any 2-clauses in S.

Every model of S consistent with $\{\overline{a}, g\}$ is consistent with b (clause c_1). Moreover, every model of S consistent with $\{\overline{a}, \overline{g}\}$ is consistent with $\{b, \delta, e\}$ (clauses c_1, c_2 and c_4). Hence, the family

$$\mathcal{A} = \{\{a\}, \{\overline{a}, g, b\}, \{\overline{a}, \overline{g}, b, \delta, e\}\}$$

is a cover for S and we set $\rho(S) = \mathcal{A}$.

All 2-clauses in $I - \{c_1\}$ are 2-clauses in $S_{\{a\}}$. Thus, $k(S_{\{a\}}) \geq k(S) - 1$. All 2-clauses in $I - \{c_1, c_4\}$ are 2-clauses in $S_{\{\overline{a},g,b\}}$. Moreover, $c_5 = e \vee \lambda$ is also a 2-clause in $S_{\{\overline{a},g,b\}}$. Since ℓ does not belong to any 2-clause in S, c_5 is not a clause of S and has no atoms in common with any 2-clause in $I - \{c_1, c_4\}$. Thus, $k(S_{\{\overline{a},g,b\}}) \geq k(S) - 1$. Finally, c_1 and c_4 are the only 2-clauses of S, which are not 2-clauses of $S_{\{\overline{a},\overline{g},b,\delta,e\}}$. Consequently, $k(S_{\{\overline{a},\overline{g},b,\delta,e\}}) \geq k(S) - 2$. Hence,

$$\Delta(S, S_A) \geq \begin{cases} 1-\alpha & \text{if } A = \{a\} \\ 3-\alpha & \text{if } A = \{\overline{a}, g, b\} \\ 5-2\alpha & \text{if } A = \{\overline{a}, \overline{g}, b, \delta, e\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1+\alpha} + \tau^{-3+\alpha} + \tau^{-5+2\alpha} = 1$$

and $\tau'_S \leq 1.67$.

Subcase (v). Exactly one of the literals d and ℓ belongs to a 2-clause in S.

Without loss of generality we will assume that S contains a 2-clause $c_5 = d \vee j$ (and consequently, ℓ does not belong to a 2-clause in S). We note that $c_2 = a \vee g \vee d$ and $c_3 = a \vee e \vee \lambda$.

Let M be a model of S. If M is consistent with $\{\overline{a}, d, e\}$, then it is consistent with $\{b\}$ (clause c_1). If M is consistent with $\{\overline{a}, d, \overline{e}\}$, it is also consistent with $\{b, g, \lambda\}$ (clauses c_1, c_4 and c_3). If M is consistent with $\{\overline{a}, \overline{d}, e\}$, it is also consistent with $\{b, j, g\}$ (clauses c_1, c_5 and c_2). Finally, if M is consistent with $\{\overline{a}, \overline{d}, \overline{e}\}$, it is consistent with $\{b, j, g, \lambda\}$ (clauses c_1, c_5, c_2 and c_3). Hence, the family

$$\mathcal{A} = \{\{a\}, \{\overline{a}, d, e, b\}, \{\overline{a}, d, \overline{e}, b, g, \lambda\}, \{\overline{a}, \overline{d}, e, b, j, g\}, \{\overline{a}, \overline{d}, \overline{e}, b, j, g, \lambda\}\}$$

is a cover for S and we define $\rho(S) = \mathcal{A}$.

All 2-clauses in $I - \{c_1\}$ are 2-clauses in $S_{\{a\}}$. Thus, $k(S_{\{a\}}) \geq k(S) - 1$. Moreover, all 2-clauses in $I - \{c_1, c_4, c_5\}$ are 2-clauses in $S_{\{\overline{a},d,e,b\}}$, $S_{\{\overline{a},d,\overline{e},b,g,\lambda\}}$, $S_{\{\overline{a},\overline{d},e,b,j,g\}}$ and $S_{\{\overline{a},\overline{d},\overline{e},b,j,g,\lambda\}}$. Hence,

$$\Delta(S, S_A) \geq \begin{cases} 1-\alpha & \text{if } A = \{a\} \\ 4-3\alpha & \text{if } A = \{\overline{a}, d, e, b\} \\ 6-3\alpha & \text{if } A = \{\overline{a}, d, \overline{e}, b, g, \lambda\}, \{\overline{a}, \overline{d}, e, b, j, g\} \\ 7-3\alpha & \text{if } A = \{\overline{a}, \overline{d}, \overline{e}, b, j, g, \lambda\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1+\alpha} + \tau^{-4+3\alpha} + 2\tau^{-6+3\alpha} + \tau^{-7+3\alpha} = 1.$$

Its root τ'_S satisfies $\tau'_S \leq 1.67$.

Subcase (vi). Both atoms d and ℓ belong to 2-clauses of S.

We can assume that d and ℓ form a 2-clause $c_5 = d \vee \ell$ as, otherwise, Case 9(iii) would apply. We note that $c_2 = a \vee g \vee d$ and $c_3 = a \vee e \vee \ell$.

We assume first that c_1, c_2 and c_3 are the only clauses containing a. We note that

$$\mathcal{A}' = \{\{\overline{a}\}, \{a, \overline{b}\}, \{a, b, \overline{g}, \overline{d}\}, \{a, b, \overline{e}, \overline{\ell}\}\}$$

is a cover for S. Indeed, let M be a minimal model of S inconsistent with every set of $\mathcal{A}'$. It follows that $a \in M$. As M is inconsistent with $\{a, \overline{b}\}$, $b \in M$. Furthermore, since M is inconsistent with $\{a, b, \overline{g}, \overline{d}\}$, $g \in M$ or $d \in M$. Similarly, $e \in M$ or $\ell \in M$. Since a belongs to the clauses c_1, c_2 and c_3 only, $M - \{a\}$ is a model of S, contradicting the minimality of M.

Let M be a model of S. If M is consistent with $\overline{a}$, it is consistent with b, as well (clause c_1). If M is consistent with $\{a, b, \overline{g}, \overline{d}\}$, it is consistent with $\{e, \ell\}$ (clauses c_4 and c_5). Lastly, if M is consistent with $\{a, b, \overline{e}, \overline{\ell}\}$, it is consistent with $\{g, d\}$ (clauses c_4 and c_5). Since $\mathcal{A}'$ is a cover for S, it follows that the family

$$\mathcal{A} = \{\{\overline{a}, b\}, \{a, \overline{b}\}, \{a, b, \overline{g}, \overline{d}, e, \ell\}, \{a, b, \overline{e}, \overline{\ell}, g, d\}\}$$

is a cover for S, as well. We set $\rho(S) = \mathcal{A}$.

All 2-clauses in $I - \{c_1\}$ are 2-clauses in $S_{\{\overline{a},b\}}$ and in $S_{\{a,\overline{b}\}}$. Thus, $k(S_{\{\overline{a},b\}}) \geq k(S) - 1$ and $k(S_{\{a,\overline{b}\}}) \geq k(S) - 1$. Moreover, all 2-clauses of $I - \{c_1, c_4, c_5\}$ are 2-clauses in $S_{\{a,b,\overline{g},\overline{d},e,\ell\}}$ and $S_{\{a,b,\overline{e},\overline{\ell},g,d\}}$. Consequently, $k(S_{\{a,b,\overline{g},\overline{d},e,\ell\}}) \geq k(S) - 3$ and $k(S_{\{a,b,\overline{e},\overline{\ell},g,d\}}) \geq k(S) - 3$. Hence,

$$\Delta(S, S_A) \geq \begin{cases} 2-\alpha & \text{if } A = \{\overline{a}, b\}, \{a, \overline{b}\} \\ 6-3\alpha & \text{if } A = \{a, b, \overline{g}, \overline{d}, e, \ell\}, \{a, b, \overline{e}, \overline{\ell}, g, d\}. \end{cases}$$

The equation (3) becomes

$$2\tau^{-2+\alpha} + 2\tau^{-6+3\alpha} = 1$$

and its root satisfies $\tau'_S \leq 1.61$.

Comment. To complete Case 9(vi) (and, in the same time, Case 9), we still need to consider a situation when S contains a 3-clause c, different from c_2 and c_3 and

such that a is an atom of c (since a appears in a 2-clause, it appears in c positively). If a is the only atom common to c and c_2 then, replacing the clause c_3 with c, we obtain a situation where Case 9(ii) or 9(iii) applies.

We can assume then that c and c_2 have at least two atoms in common. If c and c_2 have 3 atoms in common then, since each of these common atoms belongs to a 2-clause, it appears positively in every clause in S. Consequently, $c = c_2$, a contradiction. It follows that c and c_2 have exactly 2-atoms in common. Replacing c_3 by c we get a situation where Case 10(ii) applies. We will consider it below. At that point, Case 9 will be closed.

Case 10. For some atom a that occurs in a 2-clause, say c_1, there are two other clauses c_2 and c_3 that contain a and $|At(c_2) \cap At(c_3)| \geq 2$.

As before, c_2 and c_3 are 3-clauses. Throughout Case 10, we will assume that $c_1 = a \vee b$, $c_2 = a \vee \gamma \vee \delta$ and $c_3 = a \vee \epsilon \vee \lambda$. Without loss of generality, we can assume that $g = e$, that is, that γ and ϵ have the same atom.

Subcase (i). $\gamma = \overline{\epsilon}$.

As the atom g occurs negatively in S (in c_2 or c_3), g does not belong to any 2-clause.

A model of S consistent with $\{\overline{a}, \gamma\}$ is also consistent with $\{b, \lambda\}$ (clauses c_1 and c_3). Moreover, a model of S consistent with $\{\overline{a}, \overline{\gamma}\}(= \{\overline{a}, \epsilon\})$ is consistent with $\{b, \delta\}$ (clauses c_1 and c_2). Hence, the family

$$\mathcal{A} = \{\{a\}, \{\overline{a}, \gamma, b, \lambda\}, \{\overline{a}, \epsilon, b, \delta\}\}$$

is a cover for S and we set $\rho(S) = \mathcal{A}$.

All 2-clauses of $I - \{c_1\}$ are 2-clauses of $S_{\{a\}}$ and so, $k(S_{\{a\}}) \geq k(S) - 1$. All 2-clauses in $I - \{c_1\}$ that do not contain ℓ (the atom of λ) are 2-clauses in $S_{\{\overline{a},\gamma,b,\lambda\}}$. Thus, $k(S_{\{\overline{a},\gamma,b,\lambda\}}) \geq k(S) - 2$. Similarly, all 2-clauses of $I - \{c_1\}$ that do not contain d (the atom of δ) are 2-clauses in $S_{\{\overline{a},\epsilon,b,\delta\}}$ and so, $k(S_{\{\overline{a},\epsilon,b,\delta\}}) \geq k(S) - 2$. Hence,

$$\Delta(S, S_A) \geq \begin{cases} 1 - \alpha & \text{if } A = \{a\} \\ 4 - 2\alpha & \text{if } A = \{\overline{a}, \gamma, b, \lambda\}, \{\overline{a}, \epsilon, b, \delta\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1+\alpha} + 2\tau^{-4+2\alpha} = 1$$

and $\tau'_S \leq 1.65$.

Comment. We will assume in the remaining subcases of Case 10 that $|At(c_2) \cap At(c_3)| = 2$, as otherwise, Case 10(i) would apply. Indeed, let us assume that c_2 and c_3 have three atoms in common. Since $c_2 \neq c_3$, there is a literal in c_2 whose dual appears in c_3, precisely the situation covered by Case 10(i).

Subcase (ii). $\gamma = \epsilon$ and γ belongs to a 2-clause.

Clearly, in this subcase γ is an atom, that is $\gamma = g$. Let the 2-clause containing g be $c_4 = g \vee h$.

Every model of S consistent with $\{\overline{a}, g\}$ is consistent with $\{b\}$ (clause c_1). Moreover, every model of S consistent with $\{\overline{a}, \overline{g}\}$ is consistent with $\{b, h, \delta, \lambda\}$ (clauses c_1, c_4, c_2 and c_3). Hence, the family

$$\mathcal{A} = \{\{a\}, \{\overline{a}, g, b\}, \{\overline{a}, \overline{g}, b, h, \delta, \lambda\}\}$$

is a cover for S and we set $\rho(S) = \mathcal{A}$.

All 2-clauses of $I - \{c_1\}$ are 2-clauses of $S_{\{a\}}$. Thus, $k(S_{\{a\}}) \geq k(S) - 1$. All 2-clauses of $I - \{c_1, c_4\}$ are 2-clauses in $S_{\{\overline{a},g,b\}}$ and so, $k(S_{\{\overline{a},g,b\}}) \geq k(S) - 2$. Finally, all 2-clauses of $I - \{c_1, c_4\}$ that do not contain d and ℓ are 2-clauses in $S_{\{\overline{a},\overline{g},b,h,\delta,\lambda\}}$ and so, $k(S_{\{\overline{a},\overline{g},b,h,\delta,\lambda\}}) \geq k(S) - 4$. Hence,

$$\Delta(S, S_A) \geq \begin{cases} 1-\alpha & \text{if } A = \{a\} \\ 3-2\alpha & \text{if } A = \{\overline{a}, g, b\} \\ 6-4\alpha & \text{if } A = \{\overline{a}, \overline{g}, b, h, \delta, \lambda\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1+\alpha} + \tau^{-3+2\alpha} + \tau^{-6+4\alpha} = 1$$

and $\tau'_S \leq 1.67$.

Subcase (iii). $\gamma = \epsilon$, γ does not belong to any 2-clause, and at most one of d and ℓ belongs to a 2-clause.

Every model of S consistent with $\{\overline{a}, \gamma\}$ is consistent with $\{b\}$ (clause c_1). Moreover, every model consistent with $\{\overline{a}, \overline{\gamma}\}$ is consistent with $\{b, \delta, \lambda\}$ (clauses c_1, c_2 and c_3). Hence, the family

$$\mathcal{A} = \{\{a\}, \{\overline{a}, \gamma, b\}, \{\overline{a}, \overline{\gamma}, b, \delta, \lambda\}\}$$

is a cover and we set $\rho(S) = \mathcal{A}$.

All 2-clauses in $I - \{c_1\}$ are 2-clauses of $S_{\{a\}}$ and $S_{\{\overline{a},\gamma,b\}}$. Thus, $k(S_{\{a\}}) \geq k(S) - 1$ and $k(S_{\{\overline{a},\gamma,b\}}) \geq k(S) - 1$. Moreover, all 2-clauses of $I - \{c_1\}$ that do not contain d or ℓ (by our assumption, this condition excludes at most one clause) are 2-clauses in $S_{\{\overline{a},\overline{\gamma},b,\delta,\lambda\}}$ and so, $k(S_{\{\overline{a},\overline{\gamma},b,\delta,\lambda\}}) \geq k(S) - 2$. Hence,

$$\Delta(S, S_A) \geq \begin{cases} 1-\alpha & \text{if } A = \{a\} \\ 3-\alpha & \text{if } A = \{\overline{a}, \gamma, b\} \\ 5-2\alpha & \text{if } A = \{\overline{a}, \overline{\gamma}, b, \delta, \lambda\}. \end{cases}$$

the equation (3) becomes

$$\tau^{-1+\alpha} + \tau^{-3+\alpha} + \tau^{-5+2\alpha} = 1$$

and $\tau'_S \leq 1.67$.

Subcase (iv). $\gamma = \epsilon$, γ does not belong to any 2-clause and d and ℓ belong to 2-clauses of S (possibly to the same 2-clause).

Since δ and λ belong to 2-clauses, $\delta = d$ and $\lambda = \ell$.

First, we assume that c_1, c_2 and c_3 are the only clauses that contain a. Clearly, the collection

$$\mathcal{A}' = \{\{a, \overline{b}\}, \{a, b\}, \{\overline{a}, \gamma\}, \{\overline{a}, \overline{\gamma}\}\}$$

is a cover for S.

Every minimal model M of S consistent with $\{a, b\}$ is consistent with $\{\overline{\gamma}\}$ (otherwise, $M - \{a\}$ would be a model of S, too). Every model consistent with $\{\overline{a}, \gamma\}$ is consistent with $\{b\}$ (clause c_1). Finally, every model consistent with $\{\overline{a}, \overline{\gamma}\}$ is consistent with $\{b, d, \ell\}$ (clauses c_1, c_2 and c_3). Hence, the family

$$\mathcal{A} = \{\{a, \overline{b}\}, \{a, b, \overline{\gamma}\}, \{\overline{a}, \gamma, b\}, \{\overline{a}, \overline{\gamma}, b, d, \ell\}\}$$

is a cover for S and we set $\rho(S) = \mathcal{A}$.

All 2-clauses of $I - \{c_1\}$ are 2-clauses of $S_{\{a,\overline{b}\}}$, $S_{\{a,b,\overline{\gamma}\}}$ and $S_{\{\overline{a},\gamma,b\}}$. Thus, $k(S_{\{a,\overline{b}\}}) \geq k(S) - 1$, $k(S_{\{a,b,\overline{\gamma}\}}) \geq k(S) - 1$ and $k(S_{\{\overline{a},\gamma,b\}}) \geq k(S) - 1$. Moreover, all 2-clauses of $I - \{c_1\}$ that do not contain d and ℓ are 2-clauses in $S_{\{\overline{a},\overline{\gamma},b,d,\ell\}}$ and so, $k(S_{\{\overline{a},\overline{\gamma},b,d,\ell\}}) \geq k(S) - 3$. Hence,

$$\Delta(S, S_A) \geq \begin{cases} 2 - \alpha & \text{if } A = \{a, \overline{b}\} \\ 3 - \alpha & \text{if } A = \{a, b, \overline{\gamma}\}, \{\overline{a}, \gamma, b\} \\ 5 - 3\alpha & \text{if } A = \{\overline{a}, \overline{\gamma}, b, d, \ell\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-2+\alpha} + 2\tau^{-3+\alpha} + \tau^{-5+3\alpha} = 1$$

and $\tau'_S \leq 1.66$.

Let us suppose now that c_1, c_2 and c_3 are not the only clauses that contain a. Let c be a clause in S containing a and different from c_2 and c_3. Since a appears in a 2-clause, c is a 3-clause.

If $At(c) \cap At(c_2) = \{a\}$ or $At(c) \cap At(c_3) = \{a\}$, then Case 9 applies (which, we finally settled with Case 10(ii)). Thus, we can assume that c has two common atoms with c_2 and two common atoms with c_3. Since $d \neq \ell$ (otherwise, $c_2 = c_3$), there are two possibilities: (1) d and ℓ are atoms of c and, since d and ℓ appear in 2-clauses, $c = a \vee d \vee \ell$, and (2) g is the atom of c, which means that γ is a literal in c (otherwise, Case 10(i) would apply). The first possibility is covered by Case 10(ii), which applies to c_2 and c. Thus, we can assume that the second possibility holds. Let f be the atom of c other than a and g. If f does not belong to a 2-clause, then Case 10(iii) would apply to c_2 and c. Hence, we can assume that f belongs to a 2-clause and, consequently, it appears positively in all clauses in S. In particular, $c = a \vee \gamma \vee f$. Since $c \neq c_2, c_3$, $f \neq d, \ell$.

Every model of S consistent with $\{\overline{a}, \gamma\}$ is consistent with b (clause c_1). Moreover, every model of S consistent with $\{\overline{a}, \overline{\gamma}\}$ is consistent with $\{b, d, \ell, f\}$ (clauses c_1, c_2, c_3 and c). Hence, the family

$$\mathcal{A} = \{\{a\}, \{\overline{a}, \gamma, b\}, \{\overline{a}, \overline{\gamma}, b, d, \ell, f\}\}$$

is a cover for S and we set $\rho(S) = \mathcal{A}$.

All 2-clauses in $I - \{c_1\}$ are 2-clauses in $S_{\{a\}}$ and $S_{\{\overline{a},\gamma,b\}}$. Thus, $k(S_{\{a\}}) \geq k(S) - 1$ and $k(S_{\{\overline{a},\gamma,b\}}) \geq k(S) - 1$. Moreover, all 2-clauses of $I - \{c_1\}$ that do not contain d, ℓ and f are 2-clauses of $S_{\{\overline{a},\overline{\gamma},b,d,\ell,f\}}$ and so, $k(S_{\{\overline{a},\overline{\gamma},b,d,\ell,f\}}) \geq k(S) - 4$. Hence,

$$\Delta(S, S_A) \geq \begin{cases} 1 - \alpha & \text{if } A = \{a\} \\ 3 - \alpha & \text{if } A = \{\overline{a}, \gamma, b\} \\ 6 - 4\alpha & \text{if } A = \{\overline{a}, \overline{\gamma}, b, d, \ell, f\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1+\alpha} + \tau^{-3+\alpha} + \tau^{-6+4\alpha} = 1$$

and $\tau'_S \leq 1.64$.

Comment. From now on we will assume that S does not contain 2-clauses, that is, $k(S) = 0$. For an atom a, we will denote by $T(a)$ the set of 3-clauses in S with positive occurrences of a.

Case 11. There is an atom a and two 3-clauses c_1 and c_2 in $T(a)$ such that c_1 contains a literal which is dual to some literal occurring in c_2.

Without losing generality, we may assume that $c_1 = a \vee \beta \vee \gamma$ and $c_2 = a \vee \overline{\beta} \vee \delta$.

Subcase (i). $\gamma = \delta$.

Every model consistent with $\overline{a}$ is consistent with γ. Hence, the family

$$\mathcal{A} = \{\{a\}, \{\overline{a}, \gamma\}\}$$

is a cover for S and we set $\rho(S) = \mathcal{A}$. Since $k(S_A) \geq 0$, for $A \in \mathcal{A}$,

$$\Delta(S, S_A) \geq \begin{cases} 1 & \text{if } A = \{a\} \\ 2 & \text{if } A = \{\overline{a}, \gamma\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1} + \tau^{-2} = 1$$

and $\tau'_S \leq 1.62$.

Subcase (ii). There is a 3-clause c_3 in $T(a) - \{c_1, c_2\}$ such that $b \in At(c_3)$.

Without losing generality we can assume that β is a literal of c_3. Consequently, $c_3 = a \vee \beta \vee \epsilon$ and, since $c_3 \neq c_1$, $\epsilon \neq \gamma$. Moreover, if $\epsilon = \overline{\gamma}$, Case 11(i) would apply to c_1 and c_3. Thus, $\epsilon \neq \overline{\gamma}$ and $e \neq g$.

Every model of S consistent with $\{\overline{a}, \beta\}$ is consistent with δ (clause c_2). Moreover, every model of S consistent with $\{\overline{a}, \overline{\beta}\}$ is consistent with $\{\gamma, \epsilon\}$ (clauses c_1 and c_3). Hence, the family

$$\mathcal{A} = \{\{a\}, \{\overline{a}, \beta, \delta\}, \{\overline{a}, \overline{\beta}, \gamma, \epsilon\}\}$$

is a cover for S and we define $\rho(S) = \mathcal{A}$. It also follows that

$$\Delta(S, S_A) \geq \begin{cases} 1 & \text{if } A = \{a\} \\ 3 & \text{if } A = \{\overline{a}, \beta, \delta\} \\ 4 & \text{if } A = \{\overline{a}, \overline{\beta}, \gamma, \epsilon\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1} + \tau^{-3} + \tau^{-4} = 1$$

and $\tau'_S \leq 1.62$.

Comment. From now on we can assume that no clause in $T(a) - \{c_1, c_2\}$ contains b.

Subcase (iii). There is a 3-clause c_3 in $T(a) - \{c_1, c_2\}$ that contains neither d nor g.

Let $c_3 = a \vee \epsilon \vee \lambda$ (we note that c_3 does not contain b). Every model of S consistent with $\{\overline{a}, \beta\}$ is consistent with δ (clause c_2). Moreover, every model consistent with $\{\overline{a}, \overline{\beta}\}$ is consistent with γ (clause c_1). Hence, the family

$$\mathcal{A} = \{\{a\}, \{\overline{a}, \beta, \delta\}, \{\overline{a}, \overline{\beta}, \gamma\}\}$$

is a cover for S and we set $\rho(S) = \mathcal{A}$.

The theories $S_{\{\overline{a},\beta,\delta\}}$ and $S_{\{\overline{a},\overline{\beta},\gamma\}}$ contain the 2-clause $\epsilon \vee \lambda$ (it follows from the fact that e and ℓ are different from a, b, d and g). Thus, $k(S_{\{\overline{a},\beta,\delta\}}) \geq 1$ and $k(S_{\{\overline{a},\overline{\beta},\gamma\}}) \geq 1$. It follows,

$$\Delta(S, S_A) \geq \begin{cases} 1 & \text{if } A = \{a\} \\ 3+\alpha & \text{if } A = \{\overline{a},\beta,\delta\}, \{\overline{a},\overline{\beta},\gamma\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1} + 2\tau^{-3-\alpha} = 1$$

and $\tau'_S \leq 1.66$.

Comment. From now on we can assume that every clause in $T(a) - \{c_1, c_2\}$ contains either g or d.

Subcase (iv). $T(a) = \{c_1, c_2\}$.

Clearly, the collection

$$\mathcal{A}' = \{\{\overline{a},\beta\}, \{\overline{a},\overline{\beta}\}, \{a,\beta\}, \{a,\overline{\beta}\}\}$$

is a cover for S.

Every model of S consistent with $\{\overline{a},\beta\}$ is consistent with δ to satisfy c_2. Similarly, every model consistent with $\{\overline{a},\overline{\beta}\}$ is consistent with γ to satisfy c_1. Every minimal model M consistent with $\{a,\beta\}$ is consistent with $\overline{\delta}$ (otherwise $M - \{a\}$ is a model of S, as c_1 and c_2 are the only clauses in S with a positive occurrence of a). Similarly, every minimal model of S consistent with $\{a,\overline{\beta}\}$ is consistent with $\overline{\gamma}$. Hence, the family

$$\mathcal{A} = \{\{\overline{a},\beta,\delta\}, \{\overline{a},\overline{\beta},\gamma\}, \{a,\beta,\overline{\delta}\}, \{a,\overline{\beta},\overline{\gamma}\}\}$$

is a cover for S and we define $\rho(S) = \mathcal{A}$. Moreover, for every $A \in \mathcal{A}$,

$$\Delta(S, S_A) \geq 3.$$

The equation (3) becomes

$$4\tau^{-3} = 1$$

and $\tau'_S \leq 1.59$.

Comment. In the remainder of Case 11, we can assume that no two clauses in $T(a)$ have the same set of atoms. Indeed, let us assume that $c'_1, c'_2 \in T(a)$ and $At(c'_1) = At(c'_2)$. Without loss of generality, we can assume that $c'_1 = a \vee \beta' \vee \gamma'$. Then, it follows that $c'_2 = a \vee \overline{\beta'} \vee \overline{\gamma'}$ (the cases $c'_2 = a \vee \beta' \vee \overline{\gamma'}$ and $c'_2 = a \vee \overline{\beta'} \vee \gamma'$ are covered by Case 11(i)). We now note that in Cases 11(ii)-(iv) we allowed for the possibility that $\gamma = \overline{\delta}$. Thus, if $T(a) = \{c'_1, c'_2\}$, then Case 11(iv) applies. So, let us assume that there is a 3-clause $c_3 = a \vee \epsilon \vee \lambda$, such that $c_3 \in T(a) - \{c'_1, c'_2\}$. If $|\{e,\ell\} \cap \{b', g'\}| = 0$, then Case 11(iii) applies. If $|\{e,\ell\} \cap \{b', g'\}| = 1$, then Case 11(ii) applies. Finally, if $\{e,\ell\} = \{b', g'\}$, then Case 11(i) applies to c'_1 and c_3 or to c'_2 and c_3.

Subcase (v). $T(a) - \{c_1, c_2\}$ contains a clause c_3 such that at least one of $\overline{\gamma}$ and $\overline{\delta}$ is a literal of c_3.

Without loss of generality we assume that c_3 contains $\overline{\gamma}$, that is, $c_3 = a \vee \overline{\gamma} \vee \epsilon$, for some literal ϵ. We can assume that $e \neq b$ (otherwise, $At(c_1) = At(c_3)$).

Every model of S consistent with $\{\overline{a}, \overline{\beta}\}$ is consistent with $\{\gamma, \epsilon\}$ to satisfy c_1 and c_3. Moreover, every model of S consistent with $\{\overline{a}, \beta\}$ is consistent with δ to satisfy c_2. Hence, the family

$$\mathcal{A} = \{\{a\}, \{\overline{a}, \overline{\beta}, \gamma, \epsilon\}, \{\overline{a}, \beta, \delta\}\}$$

is a cover for S and we set $\rho(S) = \mathcal{A}$. Thus,

$$\Delta(S, S_A) \geq \begin{cases} 1 & \text{if } A = \{a\} \\ 4 & \text{if } A = \{\overline{a}, \overline{\beta}, \gamma, \epsilon\} \\ 3 & \text{if } A = \{\overline{a}, \beta, \delta\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1} + \tau^{-4} + \tau^{-3} = 1$$

and $\tau'_S \leq 1.62$.

Subcase (vi). No 3-clause in $T(a) - \{c_1, c_2\}$ contains $\overline{\gamma}$ or $\overline{\delta}$.

We recall that by the comment after Case 11(iii), every clause $T(a) - \{c_1, c_2\}$ contains γ or δ.

Let us now define $c_3 = a \vee \gamma \vee \delta$. We assume first that $T(a) - \{c_1, c_2\} = \{c_3\}$.

Clearly, the collection

$$\mathcal{A}' = \{\{a, \delta\}, \{a, \overline{\delta}\}, \{\overline{a}, \gamma\}, \{\overline{a}, \overline{\gamma}\}\}$$

is a cover for S.

Every minimal model M of S consistent with $\{a, \delta\}$ is consistent with $\{\overline{\gamma}, \overline{\beta}\}$. Otherwise, M would be consistent with δ and γ or with δ and β. Consequently, $M - \{a\}$ would satisfy all three clauses c_1, c_2, c_3 of $T(a)$, and since a does not belong to any other clause in S, $M - \{a\}$ would be a model of S, contrary to the minimality of M.

Moreover, every model of S consistent with $\{\overline{a}, \overline{\gamma}\}$ is consistent with $\{\beta, \delta\}$ to satisfy c_1 and c_3. Hence, the family

$$\mathcal{A} = \{\{a, \delta, \overline{\gamma}, \overline{\beta}\}, \{a, \overline{\delta}\}, \{\overline{a}, \gamma\}, \{\overline{a}, \overline{\gamma}, \beta, \delta\}\}$$

is a cover for S and we set $\rho(S) = \mathcal{A}$.

Let us observe that the theory $S_{\{\overline{a}, \gamma\}}$ contains the 2-clause $\overline{\beta} \vee \delta$ (indeed, the atoms b and d are different from a and g) and so, $k(S_{\{\overline{a}, \gamma\}}) \geq 1$. Thus,

$$\Delta(S, S_A) \geq \begin{cases} 4 & \text{if } A = \{a, \delta, \overline{\gamma}, \overline{\beta}\}, \{\overline{a}, \overline{\gamma}, \beta, \delta\} \\ 2 & \text{if } A = \{a, \overline{\delta}\} \\ 2 + \alpha & \text{if } A = \{\overline{a}, \gamma\}. \end{cases}$$

The equation (3) becomes

$$2\tau^{-4} + \tau^{-2} + \tau^{-2-\alpha} = 1$$

and $\tau'_S \leq 1.64$.

It remains to consider the case when $T(a) - \{c_1, c_2\} \neq \{c_3\}$. If $T(a) - \{c_1, c_2\} = \emptyset$, Case 11(iv) applies. Since $T(a) - \{c_1, c_2\} \neq \{c_3\}$, there is a clause $c_4 \in T(a) -$

$\{c_1, c_2\}$ such that $c_4 \neq c_3$. We can assume that $b \notin At(c_4)$ as, otherwise, Case 11(ii) would apply. Moreover, we can assume that exactly one of g and d belongs to $At(c_4)$ (if neither g nor d does, Case 11(iii) applies, and if both do, Case 11(v) applies or $c_4 = a \vee \gamma \vee \delta = c_3$, a contradiction with $c_4 \neq c_3$).

Without loss of generality, we can assume that c_4 contains γ (and so, it does not contain δ). Let $c_4 = a \vee \gamma \vee \epsilon$. Clearly, $\epsilon \neq \delta$. Moreover, $\epsilon \neq \overline{\delta}$, as no clause in $T(a) - \{c_1, c_2\}$ contains $\overline{\delta}$, by the assumption we adopted in Case 11(vi).

The family

$$\mathcal{A}' = \{\{a, \delta\}, \{a, \overline{\delta}\}, \{\overline{a}, \gamma\}, \{\overline{a}, \overline{\gamma}\}\}$$

is, trivially, a cover for S.

Let us observe that every minimal model M of S consistent with $\{a, \delta\}$ is consistent with $\overline{\gamma}$. If not, M would be consistent with γ. Since every clause of $T(a)$ contains δ or γ, $M - \{a\}$ would be a model of $T(a)$ and, consequently, of S, contrary to the minimality of M.

Moreover, every model of S consistent with $\{\overline{a}, \overline{\gamma}\}$ is consistent with $\{\epsilon, \beta, \delta\}$ to satisfy c_4, c_1 and c_2. Hence, the family

$$\mathcal{A} = \{\{a, \delta, \overline{\gamma}\}, \{a, \overline{\delta}\}, \{\overline{a}, \gamma\}, \{\overline{a}, \overline{\gamma}, \epsilon, \beta, \delta\}\}$$

is a cover for S and we define $\rho(S) = \mathcal{A}$.

We observe that the theory $S_{\{\overline{a},\gamma\}}$ contains the 2-clause $\overline{\beta} \vee \delta$ and so, $k(S_{\{\overline{a},\gamma\}}) \geq 1$. Thus,

$$\Delta(S, S_A) \geq \begin{cases} 3 & \text{if } A = \{a, \delta, \overline{\gamma}\} \\ 2 & \text{if } A = \{a, \overline{\delta}\} \\ 2 + \alpha & \text{if } A = \{\overline{a}, \gamma\} \\ 5 & \text{if } A = \{\overline{a}, \overline{\gamma}, \epsilon, \beta, \delta\}. \end{cases}$$

The the equation (3) becomes

$$\tau^{-3} + \tau^{-2} + \tau^{-2-\alpha} + \tau^{-5} = 1$$

and $\tau'_S \leq 1.66$.

Comment. From now on we will assume that for each atom a no two clauses of $T(a)$ contain dual literals. We will denote by $\Gamma(a)$ the undirected graph whose vertices are literals that belong to clauses in $T(a)$, and two literals β and γ form an edge in $\Gamma(a)$ if $a \vee \beta \vee \gamma \in S$. We will write such an edge as $\beta\gamma$. We emphasize that whenever $\beta\gamma$ is an edge of a graph $\Gamma(a)$, there is a clause $a \vee \beta \vee \gamma$ in S. By Case 3, we will assume in what follows that $\Gamma(a)$ has a nonempty set of edges.

Case 12. There is an atom a such that $\Gamma(a)$ has a vertex of degree at least 5.

Let β be a vertex of degree at least 5 in $\Gamma(a)$ and let $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ be neighbors of β in $\Gamma(a)$. In particular, it follows that S contains the clauses $a \vee \beta \vee \beta_i$, $i = 1, 2, 3, 4, 5$. Moreover, by the most recent comment, all atoms b_i, $i = 1, 2, 3, 4, 5$, are pairwise distinct.

Clearly, every model of S consistent with $\{\overline{a}, \overline{\beta}\}$ is consistent with $\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$ (clauses $a \vee \beta \vee \beta_i$, $i = 1, 2, 3, 4, 5$). Hence, the family

$$\mathcal{A} = \{\{a\}, \{\overline{a}, \beta\}, \{\overline{a}, \overline{\beta}, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}\}$$

is a cover and we set $\rho(S) = \mathcal{A}$. It follows that

$$\Delta(S, S_A) \geq \begin{cases} 1 & \text{if } A = \{a\} \\ 2 & \text{if } A = \{\overline{a}, \beta\} \\ 7 & \text{if } A = \{\overline{a}, \overline{\beta}, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1} + \tau^{-2} + \tau^{-7} = 1$$

and $\tau'_S \leq 1.66$.

Case 13. There is an atom a such that the maximum degree of a vertex in $\Gamma(a)$ is 3 or 4.

Let β be a vertex of maximum degree in $\Gamma(a)$. If the degree of β is 4 then let $\beta_1, \beta_2, \beta_3, \beta_4$ be its neighbors. Otherwise let $\beta_1, \beta_2, \beta_3$ be the neighbors of β.

Subcase (i). The degree of β is 4 and there are at least 5 edges in $\Gamma(a)$.

Let $\gamma\delta$ be an edge in $\Gamma(a)$ not incident to β. Every model of S consistent with $\{\overline{a}, \overline{\beta}\}$ is consistent with $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ (clauses $a \vee \beta \vee \beta_i$, $i = 1, 2, 3, 4$). Hence, the family

$$\mathcal{A} = \{\{a\}, \{\overline{a}, \beta\}, \{\overline{a}, \overline{\beta}, \beta_1, \beta_2, \beta_3, \beta_4\}\}$$

is a cover for S and we set $\rho(S) = \mathcal{A}$.

We observe that γ and δ are vertices in $\Gamma(a)$ and, consequently, $a \neq g, d$. Next, we note that since the edge $\gamma\delta$ is not incident to β, the atoms g, d and b are pairwise distinct. Thus, the theory $S_{\{\overline{a},\beta\}}$ contains the 2-clause $\gamma \vee \delta$ and so, $k(S_{\{\overline{a},\beta\}}) \geq 1$. It follows that

$$\Delta(S, S_A) \geq \begin{cases} 1 & \text{if } A = \{a\} \\ 2 + \alpha & \text{if } A = \{\overline{a}, \beta\} \\ 6 & \text{if } A = \{\overline{a}, \overline{\beta}, \beta_1, \beta_2, \beta_3, \beta_4\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1} + \tau^{-2-\alpha} + \tau^{-6} = 1$$

and $\tau'_S \leq 1.64$.

Subcase (ii). All edges in $\Gamma(a)$ are incident to β.

Every model consistent with $\{\overline{a}, \overline{\beta}\}$ is consistent with $\{\beta_1, \beta_2, \beta_3\}$ (clauses $a \vee \beta \vee \beta_i$, $i = 1, 2, 3$). Moreover every minimal model M of S, consistent with a, is consistent with $\overline{\beta}$, too. Otherwise, M would be consistent with β. That would imply that $M - \{a\}$ is a model of all clauses in $T(a)$ and, consequently, of S, contrary to the minimality of M. Hence, the family

$$\mathcal{A} = \{\{a, \overline{\beta}\}, \{\overline{a}, \beta\}, \{\overline{a}, \overline{\beta}, \beta_1, \beta_2, \beta_3\}\}$$

is a cover for S and we set $\rho(S) = \mathcal{A}$. Thus,

$$\Delta(S, S_A) \geq \begin{cases} 2 & \text{if } A = \{a, \overline{\beta}\}, \{\overline{a}, \beta\} \\ 5 & \text{if } A = \{\overline{a}, \overline{\beta}, \beta_1, \beta_2, \beta_3\}. \end{cases}$$

The equation (3) becomes

$$2\tau^{-2} + \tau^{-5} = 1$$

and $\tau'_S \leq 1.52$.

Comment. In the remainder of Case 13, we can assume that the degree of β is 3 and that the graph $\Gamma(a)$ has at least 4 edges. If the degree of β was 4 and $\Gamma(a)$ had 5 or more edges, Case 13(i) would apply. If the degree of β was 4 and $\Gamma(a)$ had 4 edges, or if the degree of β was 3 and $\Gamma(a)$ had 3 edges, they all would be incident to β and Case 13(ii) would apply.

Subcase (iii). The degree of β in $\Gamma(a)$ is 3 and $\Gamma(a)$ contains at least 5 edges.

It follows that $\Gamma(a)$ contains two edges, say $\gamma_1\delta_1$ and $\gamma_2\delta_2$, that are not incident to β.

We will assume first that these two edges are independent. It is easy to see that every model of S consistent with $\{\overline{a}, \overline{\beta}\}$ is consistent with $\{\beta_1, \beta_2, \beta_3\}$ (clauses $a \vee \beta \vee \beta_i$, $i = 1, 2, 3$). Hence, the family

$$\mathcal{A} = \{\{a\}, \{\overline{a}, \beta\}, \{\overline{a}, \overline{\beta}, \beta_1, \beta_2, \beta_3\}\}$$

is a cover for S and we set $\rho(S) = \mathcal{A}$.

The theory $S_{\{\overline{a},\beta\}}$ contains two 2-clauses $\gamma_1 \vee \delta_1$ and $\gamma_2 \vee \delta_2$ with disjoint sets of atoms. Consequently, $k(S_{\{\overline{a},\beta\}}) \geq 2$. Thus,

$$\Delta(S, S_A) \geq \begin{cases} 1 & \text{if } A = \{a\} \\ 2 + 2\alpha & \text{if } A = \{\overline{a}, \beta\} \\ 5 & \text{if } A = \{\overline{a}, \overline{\beta}, \beta_1, \beta_2, \beta_3\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1} + \tau^{-2-2\alpha} + \tau^{-5} = 1,$$

and $\tau'_S \leq 1.65$.

Next, we will assume that the two edges $\gamma_i\delta_i$, $i = 1, 2$, are not independent in $\Gamma(a)$. Without loss of generality we may assume that $\delta_1 = \delta_2 = \delta$.

Every model of S consistent with $\{\overline{a}, \beta, \overline{\delta}\}$ is consistent with $\{\gamma_1, \gamma_2\}$ (clauses $a \vee \gamma_1 \vee \delta$ and $a \vee \gamma_2 \vee \delta$). Moreover, every model of S consistent with $\{\overline{a}, \overline{\beta}\}$ is consistent with $\{\beta_1, \beta_2, \beta_3\}$ (clauses $a \vee \beta \vee \beta_i$, $i = 1, 2, 3$). Hence, the family

$$\mathcal{A} = \{\{a\}, \{\overline{a}, \beta, \delta\}, \{\overline{a}, \beta, \overline{\delta}, \gamma_1, \gamma_2\}, \{\overline{a}, \overline{\beta}, \beta_1, \beta_2, \beta_3\}\}$$

is a cover for S and we set $\rho(S) = \mathcal{A}$. Thus,

$$\Delta(S, S_A) \geq \begin{cases} 1 & \text{if } A = \{a\} \\ 3 & \text{if } A = \{\overline{a}, \beta, \delta\} \\ 5 & \text{if } A = \{\overline{a}, \beta, \overline{\delta}, \gamma_1, \gamma_2\}, \{\overline{a}, \overline{\beta}, \beta_1, \beta_2, \beta_3\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1} + \tau^{-3} + 2\tau^{-5} = 1$$

and $\tau'_S \leq 1.65$.

Subcase (iv). The degree of β in $\Gamma(a)$ is 3 and $\Gamma(a)$ contains exactly 4 edges.

Three of the edges of $\Gamma(a)$ are incident to β. The fourth one is not. Let us denote by $\gamma\delta$ the edge in $\Gamma(a)$ that is not incident to β.

Every model of S consistent with $\{\overline{a},\overline{\beta}\}$ is consistent with $\{\beta_1,\beta_2,\beta_3\}$ (clauses $a\vee\beta\vee\beta_i$, $i=1,2,3$). Moreover, every minimal model M of S consistent with $\{a,\beta\}$ is consistent with $\{\overline{\gamma},\overline{\delta}\}$. Otherwise, M would be consistent with at least one of γ and δ and $M-\{a\}$ would a model of $T(a)$ and so, of S, contrary to the minimality of M. Hence, the family

$$\mathcal{A}=\{\{a,\overline{\beta}\},\{a,\beta,\overline{\gamma},\overline{\delta}\},\{\overline{a},\beta\},\{\overline{a},\overline{\beta},\beta_1,\beta_2,\beta_3\}\}$$

is a cover for S and we define $\rho(S)=\mathcal{A}$.

The theory $S_{\{\overline{a},\beta\}}$ contains the 2-clause $\gamma\vee\delta$ and so, $k(S_{\{\overline{a},\beta\}})\geq 1$. Thus,

$$\Delta(S,S_A)\geq\begin{cases}2 & \text{if } A=\{a,\overline{\beta}\}\\ 4 & \text{if } A=\{a,\beta,\overline{\gamma},\overline{\delta}\}\\ 2+\alpha & \text{if } A=\{\overline{a},\beta\}\\ 5 & \text{if } A=\{\overline{a},\overline{\beta},\beta_1,\beta_2,\beta_3\}.\end{cases}$$

The equation (3) becomes

$$\tau^{-2}+\tau^{-4}+\tau^{-2-\alpha}+\tau^{-5}=1$$

and $\tau'_S\leq 1.60$.

Comment. From now on, we can assume that for every atom a, every vertex in the graph $\Gamma(a)$ has degree 1 or 2.

Case 14. There is an atom a such that $\Gamma(a)$ contains at least 4 independent edges.

Let $\gamma_1\delta_1$, $\gamma_2\delta_2$,$\gamma_3\delta_3$, $\gamma_4\delta_4$ be independent edges in $\Gamma(a)$. In this case we set $\rho(S)=\mathcal{A}$, where

$$\mathcal{A}=\{\{a\},\{\overline{a}\}\}$$

($\mathcal{A}$ is trivially complete).

The theory $S_{\{\overline{a}\}}$ contains four 2-clauses $\gamma_i\vee\delta_i$, $i=1,2,3,4$, with pairwise different atoms and so, $k(S_{\{\overline{a}\}})\geq 4$. Thus,

$$\Delta(S,S_A)\geq\begin{cases}1 & \text{if } A=\{a\}\\ 1+4\alpha & \text{if } A=\{\overline{a}\}.\end{cases}$$

The equation (3) becomes

$$\tau^{-1}+\tau^{-1-4\alpha}=1$$

and $\tau'_S=1.6701..$.

Case 15. There is an atom a such that $\Gamma(a)$ has at least 5 edges.

Subcase (i). There is a pair of nonadjacent vertices of degree 2 in $\Gamma(a)$.

Let β and γ be two nonadjacent vertices of degree 2 in $\Gamma(a)$. We denote by β_1 and β_2 the neighbors of β and by γ_1 and γ_2 the neighbors of γ.

Every model of S consistent with $\{\overline{a},\overline{\beta}\}$ is consistent with $\{\beta_1,\beta_2\}$ (clauses $a\vee\beta\vee\beta_1$ and $a\vee\beta\vee\beta_2$). Moreover, every model of S consistent with $\{\overline{a},\beta,\overline{\gamma}\}$ is consistent with $\{\gamma_1,\gamma_2\}$ (clauses $a\vee\gamma\vee\gamma_1$ and $a\vee\gamma\vee\gamma_2$). Hence, the family

$$\mathcal{A}=\{\{a\},\{\overline{a},\overline{\beta},\beta_1,\beta_2\},\{\overline{a},\beta,\gamma\},\{\overline{a},\beta,\overline{\gamma},\gamma_1,\gamma_2\}\}$$

is a cover for S and we set $\rho(S)=\mathcal{A}$.

Since the maximum degree of a vertex in $\Gamma(a)$ is 2 and $\Gamma(a)$ has at least 5 edges, there is an edge, say $\delta\epsilon$, in $\Gamma(a)$ whose endvertices are different from β, β_1 and β_2. Thus, the theory $S_{\{\overline{a},\overline{\beta},\beta_1,\beta_2\}}$ contains the 2-clause $\delta \vee \epsilon$ and so, $k(S_{\{\overline{a},\overline{\beta},\beta_1,\beta_2\}}) \geq 1$. Similarly, there is an edge, say $\lambda\varphi$, in $\Gamma(a)$ whose endvertices are different from β and γ. Hence, the theory $S_{\{\overline{a},\beta,\gamma\}}$ contains the 2-clause $\lambda \vee \varphi$ and so, $k(S_{\{\overline{a},\beta,\gamma\}}) \geq 1$.

Thus,

$$\Delta(S, S_A) \geq \begin{cases} 1 & \text{if } A = \{a\} \\ 4+\alpha & \text{if } A = \{\overline{a}, \overline{\beta}, \beta_1, \beta_2\} \\ 3+\alpha & \text{if } A = \{\overline{a}, \beta, \gamma\} \\ 5 & \text{if } A = \{\overline{a}, \beta, \overline{\gamma}, \gamma_1, \gamma_2\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1} + \tau^{-4-\alpha} + \tau^{-3-\alpha} + \tau^{-5} = 1$$

and $\tau'_S \leq 1.66$.

Subcase (ii). There are no two nonadjacent vertices of degree 2 in $\Gamma(a)$.

If there are no vertices of degree 2 in $\Gamma(a)$ then $\Gamma(a)$ contains at least 5 independent edges and Case 14 applies. Thus, let β be a vertex of degree 2 in $\Gamma(a)$ and let β_1 and β_2 be the neighbors of β.

Every model of S consistent with $\{\overline{a}, \overline{\beta}\}$ is consistent with $\{\beta_1, \beta_2\}$ (clauses $a \vee \beta \vee \beta_1$ and $a \vee \beta \vee \beta_2$). Hence, the family

$$\mathcal{A} = \{\{a\}, \{\overline{a}, \beta\}, \{\overline{a}, \overline{\beta}, \beta_1, \beta_2\}\}$$

is a cover for S and we define $\rho(S) = \mathcal{A}$.

We claim that the 3 edges in $\Gamma(a)$ that are not incident to β are independent. Indeed, let us suppose it is not so. Then some two of these edges have a common vertex, say γ. The degree of γ in $\Gamma(a)$ is 2 and none of the edges incident to γ is incident to β, contrary to the assumption we adopt in this subcase.

It follows that the theory $S_{\{\overline{a},\beta\}}$ contains three 2-clauses with pairwise disjoint sets of atoms. Consequently, $k(S_{\{\overline{a},\beta\}}) \geq 3$. Thus,

$$\Delta(S, S_A) \geq \begin{cases} 1 & \text{if } A = \{a\} \\ 2+3\alpha & \text{if } A = \{\overline{a}, \beta\} \\ 4 & \text{if } A = \{\overline{a}, \overline{\beta}, \beta_1, \beta_2\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1} + \tau^{-2-3\alpha} + \tau^{-4} = 1$$

and $\tau'_S \leq 1.67$.

Case 16. There is an atom a such that $\Gamma(a)$ has exactly 4 edges and there are two nonadjacent vertices of degree 2 in $\Gamma(a)$.

Let β and γ be two nonadjacent vertices of degree 2 in $\Gamma(a)$. We denote by β_1 and β_2 the neighbors of β and by γ_1 and γ_2 the neighbors of γ.

Clearly, the collection

$$\mathcal{A}' = \{\{a, \overline{\beta}\}, \{a, \beta\}, \{\overline{a}, \overline{\beta}\}, \{\overline{a}, \beta, \gamma\}, \{\overline{a}, \beta, \overline{\gamma}\}\}$$

is a cover for S.

Every model of S consistent with $\{\overline{a}, \overline{\beta}\}$ is consistent with $\{\beta_1, \beta_2\}$ (clauses $a \vee \beta \vee \beta_1$ and $a \vee \beta \vee \beta_2$). Moreover, every model of S consistent with $\{\overline{a}, \beta, \overline{\gamma}\}$ is consistent with $\{\gamma_1, \gamma_2\}$ (clauses $a \vee \gamma \vee \gamma_1$ and $a \vee \gamma \vee \gamma_2$). Finally, every minimal model M of S consistent with $\{a, \beta\}$ is consistent with $\{\overline{\gamma}\}$ as, otherwise, $M - \{a\}$ would be a model of S contrary to the minimality of M. Hence, the family

$$\mathcal{A} = \{\{a, \overline{\beta}\}, \{a, \beta, \overline{\gamma}\}, \{\overline{a}, \overline{\beta}, \beta_1, \beta_2\}, \{\overline{a}, \beta, \gamma\}, \{\overline{a}, \beta, \overline{\gamma}, \gamma_1, \gamma_2\}\}$$

is a cover for S and we set $\rho(S) = \mathcal{A}$. As β and γ are nonadjacent in $\Gamma(a)$, the vertices $\beta, \gamma, \gamma_1, \gamma_2$ are pairwise different. Thus,

$$\Delta(S, S_A) \geq \begin{cases} 2 & \text{if } A = \{a, \overline{\beta}\} \\ 3 & \text{if } A = \{a, \beta, \overline{\gamma}\}, \{\overline{a}, \beta, \gamma\} \\ 4 & \text{if } A = \{\overline{a}, \overline{\beta}, \beta_1, \beta_2\} \\ 5 & \text{if } A = \{\overline{a}, \beta, \overline{\gamma}, \gamma_1, \gamma_2\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-2} + 2\tau^{-3} + \tau^{-4} + \tau^{-5} = 1$$

and $\tau'_S \leq 1.67$.

Case 17. There is an atom a such that $\Gamma(a)$ has exactly 4 edges and there is exactly one vertex of degree 2 in $\Gamma(a)$.

Let β be the vertex of degree 2 in $\Gamma(a)$, let β_1 and β_2 be the neighbors of β in $\Gamma(a)$ and let $\gamma\delta$, $\epsilon\lambda$ be the two isolated edges in $\Gamma(a)$.

Every model of S consistent with $\{\overline{a}, \overline{\beta}\}$ is consistent with $\{\beta_1, \beta_2\}$ (clauses $a \vee \beta \vee \beta_1$ and $a \vee \beta \vee \beta_2$). Hence, the family

$$\mathcal{A} = \{\{a\}, \{\overline{a}, \beta\}, \{\overline{a}, \overline{\beta}, \beta_1, \beta_2\}\}$$

is a cover for S and we set $\rho(S) = \mathcal{A}$.

Both theories $S_{\{\overline{a},\beta\}}$ and $S_{\{\overline{a},\overline{\beta},\beta_1,\beta_2\}}$ contain two 2-clauses $\gamma \vee \delta$ and $\epsilon \vee \lambda$, whose sets of atoms are pairwise disjoint. Thus, $\mathrm{k}(S_{\{\overline{a},\beta\}}) \geq 2$ and $k(S_{\{\overline{a},\overline{\beta},\beta_1,\beta_2\}}) \geq 2$. Consequently,

$$\Delta(S, S_A) \geq \begin{cases} 1 & \text{if } A = \{a\} \\ 2 + 2\alpha & \text{if } A = \{\overline{a}, \beta\} \\ 4 + 2\alpha & \text{if } A = \{\overline{a}, \overline{\beta}, \beta_1, \beta_2\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1} + \tau^{-2-2\alpha} + \tau^{-4-2\alpha} = 1$$

and $\tau'_S \leq 1.67$.

Case 18. There is an atom a such that $\Gamma(a)$ contains a vertex of degree 2 and has exactly 3 edges.

Let β be a vertex of degree 2 in $\Gamma(a)$. We denote by β_1 and β_2 the neighbors of β. Let $\gamma\gamma_1$ be the edge in $\Gamma(a)$ which is not incident to β.

Clearly, the family

$$\mathcal{A}' = \{\{a, \overline{\beta}\}, \{a, \beta\}, \{\overline{a}, \overline{\beta}\}, \{\overline{a}, \beta, \gamma\}, \{\overline{a}, \beta, \overline{\gamma}\}\}$$

is a cover for S.

Every model of S consistent with $\{\overline{a}, \overline{\beta}\}$ is consistent with $\{\beta_1, \beta_2\}$ (clauses $a \vee \beta \vee \beta_1$ and $a \vee \beta \vee \beta_2$). Moreover, every model of S consistent with $\{\overline{a}, \beta, \overline{\gamma}\}$ is consistent with γ_1 (clause $a \vee \gamma \vee \gamma_1$). Finally, every minimal model M of S consistent with $\{a, \beta\}$ is consistent with $\{\overline{\gamma}, \overline{\gamma}_1\}$ as, otherwise, $M - \{a\}$ would be a model of S, contrary to the minimality of M. Hence, the family

$$\mathcal{A} = \{\{a, \overline{\beta}\}, \{a, \beta, \overline{\gamma}, \overline{\gamma}_1\}, \{\overline{a}, \overline{\beta}, \beta_1, \beta_2\}, \{\overline{a}, \beta, \gamma\}, \{\overline{a}, \beta, \overline{\gamma}, \gamma_1\}\}$$

is a cover for S and we set $\rho(S) = \mathcal{A}$. As β and γ are nonadjacent in $\Gamma(a)$, the vertices β, γ, γ_1 are pairwise different. Thus,

$$\Delta(S, S_A) \geq \begin{cases} 2 & \text{if } A = \{a, \overline{\beta}\} \\ 3 & \text{if } A = \{\overline{a}, \beta, \gamma\} \\ 4 & \text{if } A = \{a, \beta, \overline{\gamma}, \overline{\gamma}_1\}, \{\overline{a}, \overline{\beta}, \beta_1, \beta_2\}, \{\overline{a}, \beta, \overline{\gamma}, \gamma_1\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-2} + \tau^{-3} + 3\tau^{-4} = 1$$

and $\tau_S \leq 1.65$.

Case 19. There is an atom a such that the graph $\Gamma(a)$ has exactly 2 edges and they are independent.

We denote by $\beta_1\beta_2$ and $\gamma_1\gamma_2$ the two edges in $\Gamma(a)$. We define

$$\mathcal{A} = \{\{a, \overline{\beta}_1, \overline{\beta}_2\}, \{a, \overline{\gamma}_1, \overline{\gamma}_2\}, \{\overline{a}\}\}$$

Let us assume that M is a minimal model of S. If $a \notin M$ then M is consistent with $\{\overline{a}\}$. Therefore let us assume that $a \in M$. If $\beta_1, \beta_2 \notin M$ then M is consistent with $\{a, \overline{\beta}_1, \overline{\beta}_2\}$. On the other hand, if $\beta_i \in M$, for some $i = 1, 2$, then M is consistent with $\{a, \overline{\gamma}_1, \overline{\gamma}_2\}$ (otherwise $M - \{a\}$ would be a model of S, contrary to the minimality of M). It follows that the family $\mathcal{A}$ is a cover for S and we set $\rho(S) = \mathcal{A}$.

The theory $S_{\{\overline{a}\}}$ contains two 2-clauses $\beta_1 \vee \beta_2$ and $\gamma_1 \vee \gamma_2$ with pairwise disjoint sets of atoms. Consequently, $k(S_{\{\overline{a}\}}) \geq 2$. Thus,

$$\Delta(S, S_A) \geq \begin{cases} 3 & \text{if } A = \{a, \overline{\beta}_1, \overline{\beta}_2\}, \{a, \overline{\gamma}_1, \overline{\gamma}_2\} \\ 1 + 2\alpha & \text{if } A = \{\overline{a}\}. \end{cases}$$

The equation (3) becomes

$$2\tau^{-3} + \tau^{-1-2\alpha} = 1$$

and $\tau'_S \leq 1.61$.

Case 20. The graph $\Gamma(a)$ has either 1 edge or 2 adjacent edges.

Let β be any vertex of $\Gamma(a)$, if $\Gamma(a)$ has 1 edge or the vertex of degree 2, if $\Gamma(a)$ has 2 edges. We denote by γ a neighbor of β in $\Gamma(a)$.

Every minimal model M of S consistent with a is consistent with $\overline{\beta}$ (otherwise, $M - \{a\}$ would be a model of S, contrary to the minimality of M). Hence, the family

$$\mathcal{A} = \{\{\overline{a}\}, \{a, \overline{\beta}\}\}$$

is a cover for S and $\rho(S) = \mathcal{A}$. Thus,

$$\Delta(S, S_A) \geq \begin{cases} 1 & \text{if } A = \{\overline{a}\} \\ 2 & \text{if } A = \{a, \overline{\beta}\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1} + \tau^{-2} = 1$$

and $\tau'_S \leq 1.62$.

Comment. As we already noted just before Case 14, we can now assume that for every atom a, the maximum degree of a vertex in $\Gamma(a)$ is at most 2. We can also assume that for every atom a, $\Gamma(a)$ has at most 4 edges (otherwise, Case 15 applies).

Let us assume that for some atom a, $\Gamma(a)$ has at most three edges. If $\Gamma(a)$ has 1 or 2 edges, either Case 19 or Case 20 applies. Thus, we can assume that $\Gamma(a)$ has 3 edges. If $\Gamma(a)$ has a vertex of degree 2, Case 18 applies. It follows that from now on we can assume that if $\Gamma(a)$ has three edges, then it is isomorphic to the graph $3P_1$ that consists of three independent edges.

Let us now consider an atom a such that $\Gamma(a)$ contains exactly four edges. It is easy to see that the only situation not covered by Cases 16 and 17 is when $\Gamma(a)$ contains at least 2 vertices of degree 2, all of them adjacent. Thus, we can assume that $\Gamma(a)$ is isomorphic to the graph whose components are a triangle and a single isolated edge (denoted by $C_3 \cup P_1$) or the graph whose components are a 3-edge path and a single isolated edge (denoted by $P_3 \cup P_1$).

Case 21. For every atom a occurring in S, $\Gamma(a)$ is isomorphic to $C_3 \cup P_1$, $P_3 \cup P_1$ or $3P_1$. By the comment above, the assumption we adopt here covers all the situations not covered by Cases 1-20.

Subcase (i). For some atom a, there is a clause in S containing the literal $\overline{a}$.

Let $\overline{a} \vee \delta \vee \epsilon$ be a clause in S with the literal $\overline{a}$.

Let us suppose first that $\Gamma(a)$ is $P_3 \cup P_1$ or $3P_1$. Then $\Gamma(a)$ contains 3 independent edges, say $\beta_1\gamma_1$, $\beta_2\gamma_2$, $\beta_3\gamma_3$. We set $\rho(S) = \mathcal{A}$, where

$$\mathcal{A} = \{\{a\}, \{\overline{a}\}\}$$

(it is clearly a cover for S).

The theory $S_{\{a\}}$ contains a 2-clause $\delta \vee \epsilon$ and so, $k(S_{\{a\}}) \geq 1$. Moreover, the theory $S_{\{\overline{a}\}}$ contains three 2-clauses $\beta_1 \vee \gamma_1$, $\beta_2 \vee \gamma_2$ and $\beta_3 \vee \gamma_3$ with pairwise disjoint sets of atoms. Consequently, $k(S_{\{\overline{a}\}}) \geq 3$. Thus,

$$\Delta(S, S_A) \geq \begin{cases} 1 + \alpha & \text{if } A = \{a\} \\ 1 + 3\alpha & \text{if } A = \{\overline{a}\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1-\alpha} + \tau^{-1-3\alpha} = 1$$

and $\tau'_S \leq 1.66$.

Let us suppose next that $\Gamma(a)$ is the graph $C_3 \cup P_2$. Let β be a vertex of degree 2 in $\Gamma(a)$, let β_1, β_2 be the neighbors of β in $\Gamma(a)$, and let $\gamma_1\gamma_2$ be the isolated edge in $\Gamma(a)$.

Every model of S consistent with $\{\overline{a}, \overline{\beta}\}$ is consistent with $\{\beta_1, \beta_2\}$ (clauses $a \vee \beta \vee \beta_1$ and $a \vee \beta \vee \beta_2$). Hence, the family

$$\mathcal{A} = \{\{a\}, \{\overline{a}, \beta\}, \{\overline{a}, \overline{\beta}, \beta_1, \beta_2\}\}.$$

is a cover for S and we set $\rho(S) = \mathcal{A}$.

The theory $S_{\{a\}}$ contains a 2-clause $\delta \vee \epsilon$ and so, $k(S_{\{a\}}) \geq 1$. Moreover, the theory $S_{\{\overline{a},\beta\}}$ contains two 2-clauses $\beta_1 \vee \beta_2$ and $\gamma_1 \vee \gamma_2$ with pairwise disjoint sets of atoms. Consequently, $k(S_{\{\overline{a},\beta\}}) \geq 2$. Finally, the theory $S_{\{\overline{a},\overline{\beta},\beta_1,\beta_2\}}$ contains the 2-clause $\gamma_1 \vee \gamma_2$ and so, $k(S_{\{\overline{a},\overline{\beta},\beta_1,\beta_2\}}) \geq 1$. Thus,

$$\Delta(S, S_A) \geq \begin{cases} 1+\alpha & \text{if } A = \{a\} \\ 2+2\alpha & \text{if } A = \{\overline{a}, \beta\} \\ 4+\alpha & \text{if } A = \{\overline{a}, \overline{\beta}, \beta_1, \beta_2\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1-\alpha} + \tau^{-2-2\alpha} + \tau^{-4-\alpha} = 1$$

and $\tau'_S \leq 1.63$.

Subcase (ii). All occurrences of every atom in S are positive.

Let us suppose first that there is an atom a such that $\Gamma(a)$ is isomorphic to $P_3 \cup P_1$. Let d, b, c, e be the consecutive vertices of the path P_3 in $\Gamma(a)$ and let f, g be the vertices of the isolated edge in $\Gamma(a)$.

We will consider the graph $\Gamma(b)$. Clearly, it contains the edges ad and ac so it is not isomorphic to $3P_1$. The graph $\Gamma(b)$ is not isomorphic to $C_3 \cup P_1$, either. Let us suppose that it is. Then, the edge cd must be an edge of $\Gamma(b)$. Thus, S contains the clause $b \vee c \vee d$. Consequently, the graph $\Gamma(d)$ contains the edges ab and bc. The pair ac is not an edge of $\Gamma(d)$ because a belongs to 4 clauses only and $a \vee c \vee d$ is not one of them. For the same reason $\Gamma(d)$ does not contain any edge of the form ah, where $h \neq b$. Hence the graph $\Gamma(d)$ is isomorphic to $P_3 \cup P_1$ and there is an edge ch in $\Gamma(d)$, for some $h \neq a, b$. Thus, S contains the clause $c \vee d \vee h$, where $h \neq a, b$. Hence the following clauses belong to the theory S: $a \vee c \vee e$, $a \vee b \vee c$, $b \vee c \vee d$ and $c \vee d \vee h$. All of them belong to $T(c)$. Consequently, the pairs ae, ab, bd and dh belong to the graph $\Gamma(c)$. It follows that the graph $\Gamma(c)$ is connected, a contradiction with the assumptions we adopted in Case 21.

It follows that $\Gamma(b)$ is isomorphic to $P_3 \cup P_1$. Clearly, da and ac are edges of $\Gamma(b)$. Let b_1b_2, b_3b_4 be the remaining two edges of $\Gamma(b)$. Obviously, the edges b_1b_2 and b_3b_4 are independent and $b_1, b_2, b_3, b_4 \neq a$.

The graph $\Gamma(c)$ contains the edges ab and ae. Hence, $\Gamma(c)$ is isomorphic either to $C_3 \cup P_1$ or to $P_3 \cup P_1$. In both cases there is an edge, say c_1c_2, in $\Gamma(c)$ with endvertices different from a, b, e.

Clearly, the family

$$\mathcal{A}' = \{\{a, b, c\}, \{a, b, \overline{c}\}, \{a, \overline{b}\}, \{\overline{a}, b\}, \{\overline{a}, \overline{b}\}\}$$

is a cover for S.

Every model of S consistent with $\{\overline{a}, \overline{b}\}$ is consistent with $\{c, d\}$ to satisfy the clauses $a \vee b \vee c$ and $a \vee b \vee d$. Moreover, every minimal model M of S consistent

with $\{a, b, c\}$ is consistent with $\{\overline{f}, \overline{g}\}$. Otherwise $M - \{a\}$ would be a model of $T(a)$ and, consequently, of S, contrary to the minimality of M. Hence, the family

$$\mathcal{A} = \{\{a, b, c, \overline{f}, \overline{g}\}, \{a, b, \overline{c}\}, \{a, \overline{b}\}, \{\overline{a}, b\}, \{\overline{a}, \overline{b}, c, d\}\}$$

is a cover for S and we set $\rho(S) = \mathcal{A}$.

The theory $S_{\{a,b,\overline{c}\}}$ contains the 2-clause $c_1 \vee c_2$. Since $c_1, c_2 \neq a, b, c$, $k(S_{\{a,b,\overline{c}\}}) \geq 1$. The theory $S_{\{a,\overline{b}\}}$ contains two 2-clauses $b_1 \vee b_2$ and $b_3 \vee b_4$ with pairwise different atoms. Thus, $k(S_{\{a,\overline{b}\}}) \geq 2$. The theory $S_{\{\overline{a},b\}}$ contains two 2-clauses $c \vee e$ and $f \vee g$ with pairwise different atoms, so $k(S_{\{\overline{a},b\}}) \geq 2$. Finally, the theory $S_{\{\overline{a},\overline{b},c,d\}}$ contains the 2-clause $f \vee g$, so $k(S_{\{\overline{a},\overline{b},c,d\}}) \geq 1$.

Thus,

$$\Delta(S, S_A) \geq \begin{cases} 5 & \text{if } A = \{a, b, c, \overline{f}, \overline{g}\} \\ 3 + \alpha & \text{if } A = \{a, b, \overline{c}\} \\ 2 + 2\alpha & \text{if } A = \{a, \overline{b}\}, \{\overline{a}, b\} \\ 4 + \alpha & \text{if } A = \{\overline{a}, \overline{b}, c, d\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-5} + \tau^{-3-\alpha} + 2\tau^{-2-2\alpha} + \tau^{-4-\alpha} = 1$$

and $\tau_S \leq 1.66$.

Let us suppose now that there is an atom $a = a_1$ such that $\Gamma(a_1)$ is isomorphic to $C_3 \cup P_1$ and that for no atom a', $\Gamma(a')$ is isomorphic to $P_3 \cup P_1$. Let a_2, a_3, a_4 be the vertices of degree 2 in $\Gamma(a_1)$ and let b_1, c_1 be the vertices of degree 1. Clearly, $b_1, c_1 \notin \{a_1, a_2, a_3, a_4\}$. Since the graph $\Gamma(a_2)$ has a vertex of degree 2 (the edges a_1a_3 and a_1a_4 belong to $\Gamma(a_2)$), the graph $\Gamma(a_2)$ is isomorphic to $C_3 \cup P_1$ and a_3a_4 is one of its edges. It follows that $a_2 \vee a_3 \vee a_4$ is a clause in S. For the same reason $\Gamma(a_3)$ and $\Gamma(a_4)$ are isomorphic to $C_3 \cup P_1$. Let, for $i = 2, 3, 4$, b_i, c_i be the vertices of degree 1 in $\Gamma(a_i)$. Clearly, $b_i, c_i \notin \{a_1, a_2, a_3, a_4\}$, for $i = 2, 3, 4$.

If for some $i \neq j$, $\{b_i, c_i\} = \{b_j, c_j\}$, say $b_i = b_j$ and $c_i = c_j$, then pairs a_jc_i, c_ia_i are edges in $\Gamma(b_i)$ and so, the degree of c_i in $\Gamma(b_i)$ is 2. Hence $\Gamma(b_i)$ is isomorphic to $C_3 \cup P_1$ and a_ia_j is an edge in $\Gamma(b_i)$, a contradiction, as $b_i \vee a_i \vee a_j$ is not a clause in S (it follows from the fact that b_ia_j is not an edge in $\Gamma(a_i)$).

Let us assume now that $\{b_1, c_1\}$, $\{b_2, c_2\}$, $\{b_3, c_3\}$, $\{b_4, c_4\}$ are pairwise different. Suppose each pair of the sets $\{b_1, c_1\}$, $\{b_2, c_2\}$, $\{b_3, c_3\}$, $\{b_4, c_4\}$ has a common element. Then there is an element, say b_1, which belongs to all four sets. Thus, $\Gamma(b_1)$ contains the edges a_1c_1, a_2c_2, a_3c_3, a_4c_4. Hence, $\Gamma(b_1)$ is isomorphic to $C_3 \cup P_1$. Since a_1, a_2, a_3, a_4 are pairwise different and $c_i \notin \{a_1, a_2, a_3, a_4\}$, for $i = 1, 2, 3, 4$, we get $c_1 = c_2 = c_3 = c_4$ ($\Gamma(b_1)$ has 5 vertices as it is isomorphic to $C_3 \cup P_1$). This is a contradiction because we proved that $\Gamma(b_1)$ is isomorphic to a 4-edge star. Hence, some two of the sets $\{b_1, c_1\}$, $\{b_2, c_2\}$, $\{b_3, c_3\}$, $\{b_4, c_4\}$ are disjoint. We assume without loss of generality that $\{b_1, c_1\}$ and $\{b_2, c_2\}$ are disjoint.

Every model consistent with $\{\overline{a}_1, \overline{a}_2\}$ is consistent with $\{a_3, a_4\}$ to satisfy the clauses $a_1 \vee a_2 \vee a_3$ and $a_1 \vee a_2 \vee a_4$. Hence, the family

$$\mathcal{A} = \{\{a_1\}, \{\overline{a}_1, a_2\}, \{\overline{a}_1, \overline{a}_2, a_3, a_4\}\}$$

is a cover for S and we define $\rho(S) = \mathcal{A}$.

The theory $S_{\{\overline{a}_1, a_2\}}$ contains two 2-clauses with disjoint sets of atoms: $a_3 \vee a_4$ (obtained from the 3-clause $a_1 \vee a_3 \vee a_4$ in S) and $b_1 \vee c_1$ (obtained from the 3-clause $a_1 \vee b_1 \vee c_1$ in S). Hence $k(S_{\{\overline{a}_1, a_2\}}) \geq 2$. The theory $S_{\{\overline{a}_1, \overline{a}_2, a_3, a_4\}}$ also contains two 2-clauses with disjoint sets of atoms: $b_1 \vee c_1$ (obtained from the 3-clause $a_1 \vee b_1 \vee c_1$ in S) and $b_2 \vee c_2$ (obtained from the 3-clause $a_2 \vee b_2 \vee c_2$ in S). Hence $k(S_{\{\overline{a}_1, \overline{a}_2, a_3, a_4\}}) \geq 2$.

Thus,

$$\Delta(S, S_A) \geq \begin{cases} 1 & \text{if } A = \{a_1\} \\ 2 + 2\alpha & \text{if } A = \{\overline{a}_1, a_2\} \\ 4 + 2\alpha & \text{if } A = \{\overline{a}_1, \overline{a}_2, a_3, a_4\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1} + \tau^{-2-2\alpha} + \tau^{-4-2\alpha} = 1$$

and $\tau'_S \leq 1.67$.

It remains to consider the case when for every atom a in S the graph $\Gamma(a)$ is isomorphic to $3P_1$. Let b_1c_1, b_2c_2 and b_3c_3 be the edges of $\Gamma(a)$. We observe that the collection

$$\mathcal{A} = \{\{\overline{a}\}, \{a, \overline{b}_1, \overline{c}_1\}, \{a, \overline{b}_2, \overline{c}_2\}, \{a, \overline{b}_3, \overline{c}_3\}\}$$

is a cover for S and we define $\rho(S) = \mathcal{A}$. Indeed, if M is a minimal model of S such that $a \in M$ and, for each $i = 1, 2, 3$, $b_i \in M$ or $c_i \in M$, then $M - \{a\}$ would be a model of S, contrary to the minimality of M.

The theory $S_{\{\overline{a}\}}$ contains three 2-clauses $b_1 \vee c_1$, $b_2 \vee c_2$ and $b_3 \vee c_3$ with disjoint sets of atoms. Hence $k(S_{\{\overline{a}\}}) \geq 3$. Since, for every $i = 1, 2, 3$, $\Gamma(b_i)$ consists of 3 independent edges one of which is ac_i, the theory $S_{\{a, \overline{b}_i, \overline{c}_i\}}$ contains two 2-clauses with disjoint sets of atoms different from a and c_i. Hence $k(S_{\{a, \overline{b}_i, \overline{c}_i\}}) \geq 2$.

Thus,

$$\Delta(S, S_A) \geq \begin{cases} 1 + 3\alpha & \text{if } A = \{\overline{a}\} \\ 3 + 2\alpha & \text{if } A = \{a, \overline{b}_1, \overline{c}_1\}, \{a, \overline{b}_2, \overline{c}_2\}, \{a, \overline{b}_3, \overline{c}_3\}. \end{cases}$$

The equation (3) becomes

$$\tau^{-1-3\alpha} + 3\tau^{-3-2\alpha} = 1$$

and $\tau'_S \leq 1.66$.

Comment. There are no other cases to consider. Since the function ρ we described is splitting and for each S $\tau'_S \leq 1.6701..$, the Lemma 12 follows.