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Knot Choosing by Accommodating the First Derivative Constraint

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Abstract

The current cubic spline curve interpolation scheme is derived based on the implicit assumption that the magnitude of the first derivative of the curve is close to a constant. However, the assumption is not realized by the interpolation scheme. A knot choosing technique for parametric cubic spline interpolating curve construction that accommodates this assumption is presented. A comparison of the new method with several existing methods is performed and test results are included.

Keywords. *Spline curves, strain energy, smoothness, shape preserving, knots, interpolation*

1 Introduction

The problem of constructing parametric interpolating curves is of fundamental importance in computer aided geometric design/modeling, scientific computing and computer graphics. The constructed curve is often required to be smooth (satisfying some continuity requirement) and as well as 'visually pleasant'[4][9][13] (with smooth curvature distribution) while reflecting the shape suggested by the data points. Since smaller strain energy implies smoother curvature distribution, it has been popular recently to impose minimum energy constraint on the curve/surface interpolation or approximation process.

The construction of a smooth and visually pleasing parametric interpolating curve requires not only a good interpolation method, but also appropriate choice of the parameter knots. Several results have been published on knot choosing in parametric interpolation [6][8][10][12][14][17]. However, this problem is far from being completely solved yet.

In constructing parametric curves, the simplest method to choose knots is the uniform parametrization. This method usually leads to unsatisfactory result if the physical spacing of the data points are uneven. Three methods have been proposed for non-uniform parametrization, namely, chord length method, centripetal model [10] and adjusted chord length method ([5], p.111; referred as Foley's method). Experimental results show that, approximationwise, none of these methods has obvious advantage over the other ones. As far as pleasantness is concerned, centripetal model and Foley's method produce better results than the chord length method. Although these methods are widely used in constructing parametric curves, there are many occasions in which none of these methods can produce a satisfactory result. In those cases, the constructed curves using knots chosen by these methods are obviously different from the shape suggested by the data points (see Cases (a), (b) in Figures 4-5 in Section 5).

In paper[17], a new method for determining knots in parametric curve interpolation is presented (referred as ZCM method). The knots are determined using a global method. The determined knots can be used to construct interpolants which reproduce parametric quadratic curves if the interpolation scheme reproduces quadratic polynomials. When used to construct visually pleasant interpolants, the method produces curves as visually pleasing as the ones produced by the centripetal and Foley's methods. Recently, the problem for determing knots for constructing B-splines/NURBS is discussed in papers[15][16], the knots are determined using the energy-optimization method.

An important issue that has not been addressed by the current knot choosing methods for parametric cubic spline interpolation is the accommodation of the *first derivative constraint*. The current scheme for constructing a parametric cubic spline interpolating curve is derived based on the implicit assumption that the magnitude of the first derivative of the interpolating curve is close to a constant. Hence, a parametric cubic spline interpolation process should ensure that the constraint is accommodated by the interpolating spline curve. Unfortunately, no such effort has been found in the literature yet. An interpolating curve construction process that does not satisfy this constraint may result

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in a curve that is not visually pleasing at all (see cases (a) and (b) of Figure 4). In this paper we will present a knot choosing method for the parametric cubic spline interpolation process that accommodates this constraint. Test results show that the new method produces interpolating curves visually more pleasant than the chord length method, the centripetal model , Foley's method and the ZCM method in most of the cases.

The remaining part of the paper is arranged as follows. The basic idea of the new method is described in section 2. The idea of choosing knots in constructing a quadratic interpolating curve is studied in Section 3. Based on the discussion of Sections 2 and 3, a new method for knot choosing in constructing visually pleasant cubic spline curves is presented in Section 4. Implementation and test results of the new method with current methods on several representative data sets are given in Section 5. Concluding remarks are given in Section 6.

2 Basic idea

Let $P_i = (x_i, y_i)$, $1 \le i \le n$, be a set of data points satisfying the condition $P_i \neq P_{i+1}$ for all i. Our goal is to construct a parametric curve interpolant to P_i, $i = 1, 2, ..., n$, with a visually pleasant shape suggested by the data points. $P(t)$ is usually constructed as a piecewise parametric curve, with a cubic spline curve as the main choice. If $P(t)$ is constructed as a cubic spline, then on each interval $[t_i, t_{i+1}]$, $i = 1, 2, ..., n-1$, $P(t)$ may be defined as follows:

$$P_i(t) = \varphi_0(s)P_i + \varphi_1(s)h_iM_i + \psi_1(s)h_iM_{i+1} + \psi_0(s)P_{i+1}, \tag{1}$$

where

$$\varphi_0(s) = (s-1)^2(2s+1), \quad \varphi_1(s) = (s-1)^2 s,$$
$$\psi_0(s) = s^2(-2s+3), \quad \psi_1(s) = s^2(s-1)$$

are cubic Hermite basis functions, $s = (t - t_i)/h_i$ with $h_i = t_{i+1} - t_i$, and M_i are the derivatives of $P(t)$ at t_i. It is generally accepted that $P(t)$ would have a satisfactory shape if M_i are determined by minimizing the internal strain energy of the spline curve

$$\int [\frac{d^2\kappa(\varpi)}{d\varpi^2}]^2 d\varpi, \tag{2}$$

where $\kappa(\varpi)$ is the curvature of the spline $P(t)$, defined as a function of the arc length of $P(t)$.

If the magnitude of the first derivative of $P(t)$ is close to a constant C, then

$$\varpi = \int_{t_0}^{t} |P(t)^{'}|dt \approx C(t - t_0),$$

and (2) can approximated by the following simple form

$$\frac{1}{C}\int |P^{''}(t)|^2 dt. \tag{3}$$

Consequently, we have the following system of $n-2$ equations for $i = 2, ..., n-1$,

$$\begin{aligned} &h_iM_{i-1} + (h_{i-1} + h_i)M_i + h_iM_{i+1} = \\ &\frac{3h_i}{h_{i-1}}(P_i - P_{i-1}) + \frac{3h_{i-1}}{h_i}(P_{i+1} - P_i). \end{aligned} \tag{4}$$

$P(t)$ is determined by solving eq. (4) with two end conditions at P_1 and P_n. The only quantities that can change the shape of $P(t)$, consequently, are h_i, or, the knots t_i, $i = 1, 2, \cdots, n$. Obviously these knots should be chosen in a way so that the above first derivative constraint is indeed satisfied by the interpolating curve. The basic idea of a knot choosing technique that accommodates this first derivative constraint is shown below.

Let t_{i-1}, t_i and t_{i+1} be three knots corresponding to P_{i-1}, P_i and P_{i+1} respectively. The degree two parametric Lagrange polynomial $Q_i(t)$ which interpolates P_{i-1}, P_i and P_{i+1} at t_{i-1}, t_i and t_{i+1}, respectively, is defined by

$$\begin{aligned} Q_i(t) = &\frac{(t-t_i)(t-t_{i+1})}{(t_{i-1}-t_i)(t_{i-1}-t_{i+1})}(P_{i-1} - P_i) + \\ &\frac{(t-t_{i-1})(t-t_i)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)}(P_{i+1} - P_i) + P_i. \end{aligned} \tag{5}$$

To simplify the process of making the magnitude of the first derivative of $P(t)$ close to a constant, the segment of $P(t)$ between P_{i-1} and P_{i+1} will be approximated by $Q_i(t)$. Hence, making the magnitude of the first derivative of $P(t)$ close to a constant is essentially a process of making the magnitude of the first derivative of $Q_i(t)$ close to a constant.

By substituting

$$\begin{aligned} t &= t_{i-1} + (t_{i+1} - t_{i-1})s \\ s_i &= (t_i - t_{i-1})/(t_{i+1} - t_{i-1}) \end{aligned} \tag{6}$$

into (5) one obtains

$$\begin{aligned} Q_i(s) = &\frac{(s-s_i)(s-1)}{s_i}(P_{i-1} - P_i) + \\ &\frac{s(s-s_i)}{1-s_i}(P_{i+1} - P_i) + P_i, \qquad 0 \le s \le 1, \end{aligned} \tag{7}$$

where s_i is a variable to be determined between 0 and 1.

With s_i being a variable, $Q_i(s)$ is actually a family of curves. The idea here is to choose an s_i that would make *the magnitude of the first derivative of $Q_i(s)$ close to a constant.* Let $Q_i(s) = (x_i(s), y_i(s))$, and c_x and c_y be two constants. This goal may be realized by minimizing the following objective function:

$$f(s_i) = \int_0^1 [(\frac{dx_i(s)}{ds} - c_x)^2 + (\frac{dy_i(s)}{ds} - c_y)^2] ds.$$

Details of the process are shown in the subsequent sections.

3 Determining Free Variable s_i

Straightforward computation shows that

$$f(s_i) = \frac{\Delta_{i-1}^2}{3}(\frac{1}{s_i^2} - \frac{2cos\alpha_i}{s_i(1-s_i)}\Delta_{i-1}\Delta_i + \frac{\Delta_i^2}{(1-s_i)^2}) + D_i,$$

where

$$D_i = (x_{i+1} - x_{i-1} - c_x)^2 + (y_{i+1} - y_{i-1} - c_y)^2,$$

$\Delta_{i-1} = |P_{i-1}P_i|$ denotes the distance between P_{i-1} and P_i, and α_i is the angle between the vectors $P_{i-1}P_i$ and P_iP_{i+1}, as shown in Figure 1.

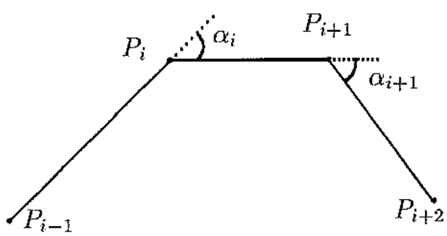

Figure 1. Angle α_i

The free variable s_i is determined by the following equation

$$\frac{df(s_i)}{ds_i} = 0. \quad (8)$$

We discuss solutions of (8) for three special cases $\alpha_i = 0$, $\alpha_i = \frac{\pi}{2}$, and $\alpha_i = \pi$ first.

a) $\alpha_i = 0$: $\cos\alpha_i = 1$, i.e., P_i is a point between P_{i-1} and P_{i+1}. Then

$$f(s_i) = \frac{1}{3}(\frac{1}{s_i}\Delta_{i-1} - \frac{1}{1-s_i}\Delta_i)^2 + D_i.$$

By defining

$$s_i = \frac{\Delta_{i-1}}{\Delta_{i-1} + \Delta_i}, \quad (9)$$

we have $f(s_i) = D_i$, and (8) is satisfied. Substituting (9) into $dQ_i(s)/ds$, one gets $|dQ_i(s)/ds| = \Delta_{i-1} + \Delta_i$. This means that if P_i is a point between P_{i-1} and P_{i+1}, and s_i is defined by (9), then the magnitude of the first direvative of $Q_i(s)$ is a constant and Q_i is a straight line that is the most pleasant curve one can get in this case.

b) $\alpha_i = \frac{\pi}{2}$: $\cos\alpha_i = 0$, then (8) becomes

$$(1-s_i)^3\Delta_{i-1}^2 - s_i^3\Delta_i^2 = 0.$$

Thus

$$s_i = \frac{\Delta_{i-1}^{\frac{2}{3}}}{\Delta_{i-1}^{\frac{2}{3}} + \Delta_i^{\frac{2}{3}}}. \quad (10)$$

c) $\alpha_i = \pi$: $cos\alpha_i = -1$, i.e., vectors $P_{i-1}P_i$ and P_iP_{i+1} are in opposite directions. Then

$$f(s_i) = \frac{1}{3}(\frac{1}{s_i}\Delta_{i-1} + \frac{1}{1-s_i}\Delta_i)^2 + D_i.$$

The solution of (8) is

$$s_i = \frac{\Delta_{i-1}^{\frac{1}{2}}}{\Delta_{i-1}^{\frac{1}{2}} + \Delta_i^{\frac{1}{2}}}. \quad (11)$$

Substituting this solution into $dQ_i(s)/ds$, one gets $|dQ_i(s_i)/ds| = 0$. This means that if vectors $P_{i-1}P_i$ and P_iP_{i+1} are of opposite direction, and s_i is defined by (11), then the magnitude of the first direvative of $Q_i(s)$ at s_i is zero. If $Q_i(s)$ is viewed as the trajectory of a particle, then $|dQ_i(s_i)/ds| = 0$ means that the speed of the particle is zero when passing through the point P_i. The zero speed makes it easy for the particle to turn around at P_i. It is easy to see that in this case the trajectory of $Q_i(s)$ is the polygon $P_{i-1}P_iP_{i+1}$, and this is the most pleasant trajectory one can get.

In the following we will use the values of s_i at $\alpha_i = 0$, $\pi/2$, and π to define the value of s_i for $0 \leq \alpha_i \leq \pi$.

Suppose that $t_{i+1} - t_i$ can be expressed as a function of α_i, i.e., $t_{i+1} - t_i = f(\alpha_i)$, then $t_{i+1} - t_i$ can be expanded as a Taylor series at $\alpha_i = 0$

$$t_{i+1} - t_i = f(0) + \frac{df(0)}{d\alpha_i} * \alpha_i + O(\alpha_i^2) \quad (12)$$

For $\alpha_i = 0$, i.e., P_{i-1}, P_i and P_{i+1} are on the same line in that order, setting $t_i - t_{i-1} = \Delta_{i-1}$ and $t_{i+1} - t_i = \Delta_i$ is a natural choice, according to eqs. (6) and (9), thus we have $f(0) = \Delta_i$. Since $t_{i+1} - t_i = \Delta_i$ and $t_i - t_{i-1} = \Delta_{i-1}$ at $\alpha_i = 0$, it follows from eq. (12) that, for $0 \leq \alpha_i \leq \pi$, $t_i - t_{i-1}$ and $t_{i+1} - t_i$ can be approximated by the first and second terms of their taylor expansions as follows:

$$\begin{aligned} t_i - t_{i-1} &= \Delta_{i-1}(1 + b_i\alpha_i/\pi), \\ t_{i+1} - t_i &= \Delta_i(1 + a_i\alpha_i/\pi). \end{aligned} \quad (13)$$

where b_i and a_i are to be determined. Note that (13) satisfies (9) for $\alpha_i = 0$.

By substituting $\alpha_i = \pi/2$ and π into (10), (11) and (13), one gets

$$\frac{\Delta_{i-1}(1 + b_i/2)}{\Delta_{i-1}(1 + b_i/2) + \Delta_i(1 + a_i/2)} = \frac{\Delta_{i-1}^{\frac{2}{3}}}{\Delta_{i-1}^{\frac{2}{3}} + \Delta_i^{\frac{2}{3}}},$$

$$\frac{\Delta_{i-1}(1 + b_i)}{\Delta_{i-1}(1 + b_i) + \Delta_i(1 + a_i)} = \frac{\Delta_{i-1}^{\frac{1}{2}}}{\Delta_{i-1}^{\frac{1}{2}} + \Delta_i^{\frac{1}{2}}}.$$

The solutions are

$$b_i = (\frac{\Delta_i}{\Delta_{i-1}})^{\frac{1}{3}} + (\frac{\Delta_i}{\Delta_{i-1}})^{\frac{1}{6}} - 1,$$
$$a_i = (\frac{\Delta_{i-1}}{\Delta_i})^{\frac{1}{3}} + (\frac{\Delta_{i-1}}{\Delta_i})^{\frac{1}{6}} - 1. \quad (14)$$

Hence, for general $0 \leq \alpha_i \leq \pi$, s_i is defined as

$$s_i = \frac{\Delta_{i-1}(\pi + b_i\alpha_i)}{\Delta_{i-1}(\pi + b_i\alpha_i) + \Delta_i(\pi + a_i\alpha_i)} \quad (15)$$

with a_i and b_i being defined in (14). This definition of s_i satisfies eqs. (9), (10) and (11).

4 Determining Knots t_i

We will determine the values of the knots t_i in this section.

From the second case in (13) one gets an approximated value of $t_{i+1} - t_i$. By working with α_{i+1} using P_i, P_{i+1} and P_{i+2}, one gets another approximated value of $t_{i+1} - t_i$ (see case one of (13) with i replaced with $i + 1$). These values are shown below.

$$t_{i+1} - t_i = \Delta_i(1 + a_i\alpha_i/\pi),$$
$$t_{i+1} - t_i = \Delta_i(1 + b_{i+1}\alpha_{i+1}/\pi).$$

It follows from eqs. (12) and (13) that to get more precision result, $t_{i+1} - t_i$ can be expressed as

$$t_{i+1} - t_i = \Delta_i(1 + A_i a_i\alpha_i/\pi),$$
$$t_{i+1} - t_i = \Delta_i(1 + B_i b_{i+1}\alpha_{i+1}/\pi).$$

where A_i and B_i are unkowns to be determined.

These two expressions can be used to define $t_{i+1} - t_i$. Since the formation of $t_{i+1} - t_i$ is related to both α_i and α_{i+1}, $t_{i+1} - t_i$ should be defined as a combination of these two expressions. By testing many sets of data points, the following combination (16), based on the lengths of the adjacent legs of P_iP_{i+1}, has been proven to be is a better choice to define $t_{i+1} - t_i$

$$t_{i+1} - t_i = \Delta_i(1 + \frac{\lambda_i^2 a_i\alpha_i + \gamma_i^2 b_{i+1}\alpha_{i+1}}{\lambda_i + \gamma_i}) \quad (16)$$

where $A_i = \lambda_i$ and $B_i = \gamma_i$

$$\lambda_i = 1 + \frac{\Delta_{i-1}}{\Delta_i}, \quad \gamma_i = 1 + \frac{\Delta_{i+1}}{\Delta_i}.$$

The factors λ_i and γ_i have the property that if the value of $a_i\alpha_i$ is bigger than that of $b_{i+1}\alpha_{i+1}$, then $a_i\alpha_i$ has a bigger influence on the formation of $t_{i+1} - t_i$ than $b_{i+1}\alpha_{i+1}$. Eq. (16) holds for $i = 1, 2, ..., n - 1$ with $t_1 = \alpha_1 = \alpha_n = \Delta_0 = \Delta_{n+1} = 0$.

For evenly spaced data points, knots defined by (16) generally produce better results when applied to cubic spline interpolation. Several examples are shown in Figure 2. The data points used to produce the curves are defined by

$$\begin{aligned} &P_3 = (0, 0), \ P_4 = (1, 0), \\ &P_1 = P_3 + (2cos(\pi - \alpha_3), 2sin(\pi - \alpha_3)), \\ &P_2 = (P_1 + \phi P_3)/(1 + \phi), \\ &P_6 = P_4 + (2cos(\alpha_4), 2sin(\alpha_4)), \\ &P_5 = (P_4/\phi + P_6)/(1/\phi + 1). \end{aligned} \quad (17)$$

The curves shown in Figure 2 are cubic spline interpolating curves of these data points with ϕ in (17) set to 1 and varying α_3 and α_4 values. The symbol "+" denotes the position of a given data point.

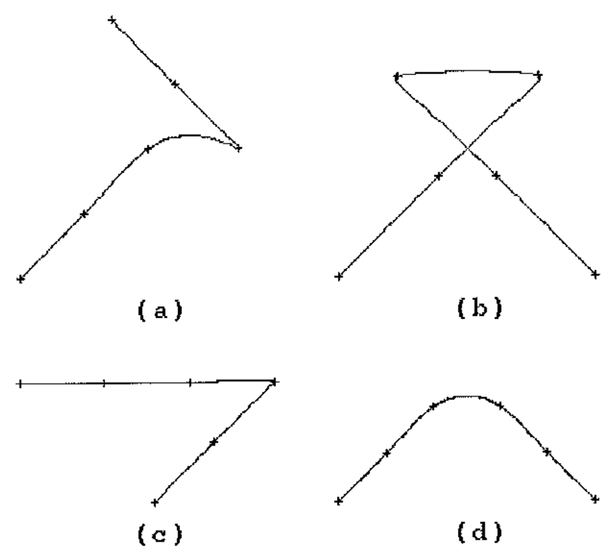

Figure 2. **(a)** $\alpha_3 = \pi/4$, $\alpha_4 = 3\pi/4$, **(b)** $\alpha_3 = 3\pi/4$, $\alpha_4 = -3\pi/4$, **(c)** $\alpha_3 = 0$, $\alpha_4 = -3\pi/4$, **(d)** $\alpha_3 = \pi/4$, $\alpha_4 = -\pi/4$

The experiments show that with the value of ϕ in (17) set to 1, the knots produced by (16) produce satisfactory cubic spline interpolating curves for all α_3 and α_4 satisfying $0 \leq \alpha_3 \leq 3\pi/4$ and $-3\pi/4 \leq \alpha_4 \leq 3\pi/4$. It should be pointed out that Foley's method produces satisfactory interpolating curves for all the α_3 and α_4 tested as well. This is because Foley's method also considers the influence of angles between adjacent control legs in the construction of t_i.

For unevenly spaced data points, the values of some Δ_{i-1}/Δ_i and Δ_{i+1}/Δ_i could become large and, consequently, lead to large $t_{i+1} - t_i$ and unsatisfactory results. For instance, if the coordinates of the given data points are (0,0), (26,24), (28,24), and (54,0) [10], then the knots determined by (16) are 0.0, 29.3, 70.9, 100.3, respectively. The cubic spline curve constructed with these knots has a loop between (26,24) and (28,24). This shortcoming can be overcome by putting a constraint on the magnitude of the second and third items in (16), as follows.

$$t_{i+1} - t_i = \begin{cases} \Delta_i(1+\tau_i) & if \quad \tau_i \le \kappa_i, \\ \Delta_i(1+\kappa_i + \dfrac{\tau_i - \kappa_i}{1+\tau_i-\kappa_i}), & otherwise, \end{cases} \qquad (18)$$

with

$$\tau_i = (\lambda_i{}^2 a_i \alpha_i + \gamma_i{}^2 b_{i+1}\alpha_{i+1})/(\lambda_i + \gamma_i),$$
$$\kappa_i = \mu_i + \nu_i + \sqrt{\alpha_i \alpha_{i+1}},$$

$$\mu_i = \frac{\Delta_{i-1}}{\Delta_{i-1}+\Delta_i}, \quad \nu_i = \frac{\Delta_{i+1}}{\Delta_i + \Delta_{i+1}}.$$

where λ_i and γ_i are defined in (16). Note that the maximum value of $\kappa_i + (\tau_i - \kappa_i)/(1+\tau_i-\kappa_i)$ is $3+\pi$, which is an approximation of 2π, the maximum value of the second and third items in (16) for equally spaced data points.

Experiments showed that (18) gives better results for the data points defined by (17) for all ϕ, α_3 and α_4, which satisfy $1/3 \le \phi \le 3$, $0 \le \alpha_3 \le 3\pi/4$ and $-3\pi/4 \le \alpha_4 \le 3\pi/4$.

5 Experiments

Four sets of representative data points have been used to compare the new method with the centripetal, Foley's and ZCM methods. The uniform and chord length methods are not included in the comparison as they produce unsatisfactory curves for all the four sets of representative data points. The comparison is performed using knots determined by these methods in the construction of a parametric cubic spline curve which interpolates the given data points.

The three-point difference formula [3] is used to determine the end conditions of the spline curve. The four sets of representative data points are the Akima [1], FRN 15A [7], Brodlie [2] and Lee [10] data points. The curves generated by these methods are shown in Figures 3-6, where the symbol '+' denotes the location of a given data point. Figures 3-6 show that the curves generated by the new method possess the shape suggested by the data points, and are more visually pleasing than the ones generated by centripetal model and Foley's method. Figures 3-6 also show that the ZCM method and the new method produce similar visually pleasing curves for Akima, FRN 15A and Lee data points, but different curves for Brodlie data points. Another difference of these two methods is that ZCM method is global, whereas the new one is local. Using the four sets of representative data points, we have also compared the four methods on natural cubic splines and the results are basically the same as the ones shown in Figures 3-6.

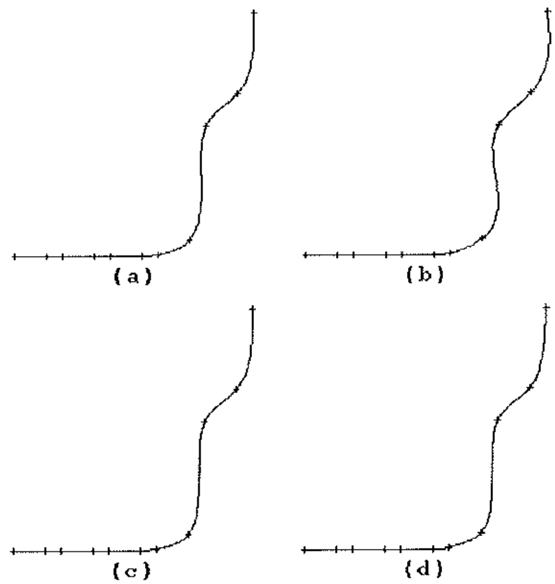

Figure 3. Data points in (Akima, 1970), {x,y}={(0,10), (2,10), (3,10), (5,10), (6,10), (8,10), (9,10.5), (11,15), (12,50), (14,60), (15,85)}. (a) centripetal model , (b) Foley's method, (c) ZCM method, (d) new method.

6 Conclusions

A new knot choosing technique for parametric cubic spline interpolation process is presented. The new cubic spline curve interpolation process makes more sense because it is derived based on the first derivative constraint, i.e., on each interval, the knots are chosen by requiring the magnitude of the first derivative of the two adjacent curve segments to be close to a constant.

Test results comparing the new method with the centripetal model , Foley's method and the ZCM method show that 1) when used to construct parametric cubic spline curves, the new method, the centripetal model , Foley's method and the ZCM method generally produce better results than the chord length method; 2) in most of the cases the new method produces more visually pleasant curves than the centripetal model , Foley's method and the ZCM method.

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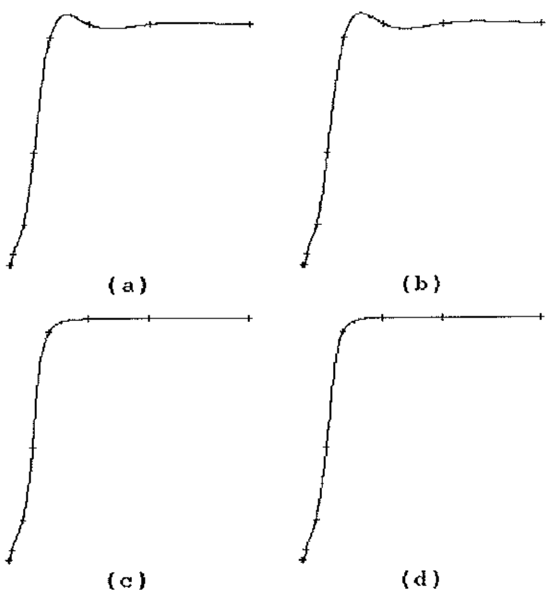

Figure 4. Data points in (Fritsch and Carlson, 1980), {x,y}={(7.99,0), (8.09,2.76429e-5), (8.19,4.37498e-2), (8.7,0.169183), (9.2,0.469428), (10,0.94374), (12,0.998636), (15,0.999919), (20,0.999994)}. (a) centripetal model , (b) Foley's method, (c) ZCM method, (d) new method.

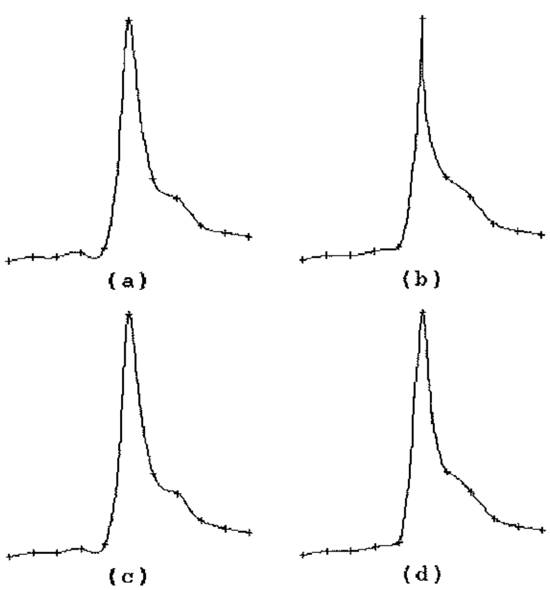

Figure 5. Data points in (Brodlie, 1980), {x,y}={(0,1), (1,1.1), (2,1.1), (3,1.2),(4,1.3), (5,7.2),(6,3.1), (7,2.6), (8,1.9), (9,1.7), (10,1.6)}. (a) centripetal model , (b) Foley's method, (c) ZCM method, (d) new method.

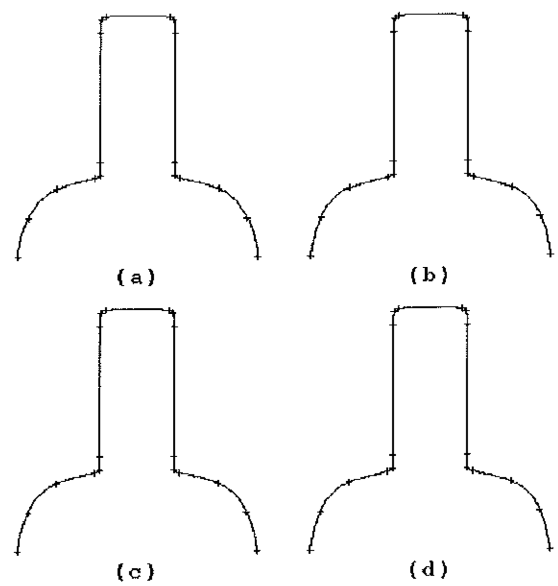

Figure 6. Data points in (Lee, 1989), {x,y}={(0,0), (1.34,5), (5,8.66), (10,10), (10.6,10.4), (10.7,12), (10.7,28.6), (10.8,30.2), 11.4,30.6), (19.6,30.6), (20.2,30.2)}. (a) centripetal model , (b) Foley's method, (c) ZCM method, (d) new method.

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