Conversion from the Doo-Sabin to Catmull-Clark Subdivision Surfaces

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Abstract

Subdivision surfaces are powerful tools for graphical modeling and animation because of their scalability, numerical stability, simplicity in coding and, especially, their ability to represent complex shape of arbitrary topology.

The Doo-Sabin and Catmull-Clark surfaces are two popular subdivision surfaces used in graphics community. They are constructed by generalizing the idea of obtaining uniform biquadratic and uniform bicubic B-spline patches from a rectangular mesh, respectively.

In this paper, we make clear the relationship between the Doo-Sabin and Catmull-Clark surfaces and show how to convert the Doo-Sabin to Catmull-Clark surfaces by elevating the degree of the original surface by one. Based on our method, the rectangular faces of the usual uniform Doo-Sabin surface are exactly converted to those of a non-uniform Catmull-Clark surface: one of the NURSS(non-unifrom recursive subdivision surface) and the other faces are approximately done to those of it.

Keywords: Doo-Sabin subdivision surface, Catmull-Clark subdivision surface, NURSS(Non-uniform Recursive subdivision surface), Degree elevation.

1 Introduction

Subdivision surfaces are powerful tools for graphical modeling and animation because of their scalability, numerical stability, simplicity in coding and, especially, their ability to represent complex shape of arbitrary topology.

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In this paper, we make clear the relationship between the Doo-Sabin and Catmull-Clark surfaces and show how to convert the Doo-Sabin to Catmull-Clark surfaces by elevating the degree of the original surface by one. In fact, the usual uniform Doo-Sabin surface is approximately converted to a non-uniform Catmull-Clark surface: one of the NURSS(non-unifrom recursive subdivision surface).

2 Doo-Sabin and Catmull-Clark Subdivisions

In this section we will review the Doo-Sabin and Catmull-Clark surfaces as well as the Chaikin's algorithm and the B-spline subdivision curves that triggered the developments of the two surfaces.

2.1 Chaikin's algorithm

Chaikin's algorithm [2] generates a quadratic B-spline curve from a polygon by successively cutting its corners. Each subdivision generates two new points on each polygon leg at $(1/4, 3/4)$. For a polygon with $n+1$ vertices $\boldsymbol{p}_i^j$, $i = 0, ..., n$ at the subdivision depth j, two new points defined as follows are inserted into the polygon leg $\boldsymbol{p}_i^j \boldsymbol{p}_{i+1}^j$

$$\boldsymbol{p}_{2i}^{j+1} := \frac{3}{4}\boldsymbol{p}_i^j + \frac{1}{4}\boldsymbol{p}_{i+1}^j, \qquad (1)$$

$$\boldsymbol{p}_{2i+1}^{j+1} := \frac{1}{4}\boldsymbol{p}_i^j + \frac{3}{4}\boldsymbol{p}_{i+1}^j. \qquad (2)$$

2.2 B-spline Subdivision

B-spline subdivision is a generalization of Chaikin's algorithm and it obeys a refinement equation [11]. The refinement equation for B-splines of degree l is given by

$$B_l(t) = \frac{1}{2^l}\sum_{k=0}^{l+1}\begin{pmatrix} l+1 \\ k \end{pmatrix} B_l(2t-k). \qquad (3)$$

Quadratic B-spline subdivision is identical to Chaikin's algorithm and cubic B-spline subdivision is given by

$$\boldsymbol{p}_{2i}^{j+1} := \frac{1}{8}(\boldsymbol{p}_i^j + 6\boldsymbol{p}_{i+1}^j + \boldsymbol{p}_{i+2}^j), \qquad (4)$$

$$\boldsymbol{p}_{2i+1}^{j+1} := \frac{1}{2}(\boldsymbol{p}_{i+1}^j + \boldsymbol{p}_{i+2}^j). \qquad (5)$$

2.3 Doo-Sabin surface

Doo and Sabin [4] extended Chaikin's idea for curves to generate surfaces. A surfaces is generated from a polyhedral network by successively cutting off its corners and edges. An algorithm may be given as follows [9]:

1. For every vertex V_i^j of the polyhedron P^j, a new vertex V_i^{j+1}, called an *image*, is generated on each face adjacent to V_i^j.

2. For each face F_i^j of P^j, a new face, called an *F-face*, is constructed by connecting the *images* vertices V_i^{j+1} generated in Step 1.

3. For each edge E_i common to two faces F_k^j and F_l^j, a new 4-sided face, called an *E-face*, is constructed by connecting the *images* of the end vertices of E_i^j on the faces F_k^j and F_l^{j+1}.

4. For each vertex V_i^j, where n faces meet, a new face, called a *V-face*, is constructed by connecting the images of V_i^j on the faces meeting at V_i^j.

An *image* vertex V_i^{j+1} generated in Step 1 depends only on the vertices of P^j and is given by

$$V_i^{j+1} := \sum_{k=0}^{n-1} a_{ik} V_k^j \qquad (6)$$

where V_k^j are vertices of the old faces and a_{ij} are coefficients defined as follows:

$$a_{ik} = \frac{n+5}{4n}, \quad \text{for } i = k,$$

$$a_{ik} = \frac{3 + 2\cos(2\pi(\frac{i-k}{n}))}{4n}, \quad \text{for } i \neq k. \qquad (7)$$

2.4 Catmull-Clark surface

The Catmull-Clark subdivision method [1] is similar to the Doo-Sabin method, but is based on the tensor product bicubic splines. It produces surfaces that are C^2 everywhere except at extraordinary vertices, where they are C^1.

The rules of the subdivision for a n-sided face are as follows[1]:

1. New face points are calculated as the average of all of the old points defining the face.

2. New edge points are obtained as the average of all of the midpoints of the old edge with the average of the two new face points.

3. New vertex points are given by the average:

$$\frac{Q}{n} + \frac{2R}{n} + \frac{S(n-3)}{n} \qquad (8)$$

   where Q is the average of the new face points of all faces adjacent to the old vertex point. R is the average of the midpoints of all old edges incident on the old vertex point and S is the old vertex point.

After these points have been computed, new edges are formed by

1. connecting each new face point to the new edge points of the edges defining the old faces.

2. connecting each new vertex point to the new edge points of all old edges incident on the old vertex point.

New faces are then defined as those enclosed by new edges.

3 Conversion from quadratic to cubic subdivision curves

In the previous section, we have mentioned that the Doo-Sabin and Catmull-Clark surfaces are based on the tensor product biquadratic and bicubic splines, respectively. In this section, before we discuss how to convert the Doo-Sabin to Catmull-Clark surfaces, we will argue how to convert a quadratic to cubic subdivision curves. The conversion process is usually called as the degree elevation. The conversion process from the Doo-Sabin to Catmull-Clark surfaces can be regarded as the extension of the curve conversion to the surface. Therefore it can be said to be a kind of the degree elevation.

3.1 Degree elevation

The degree elevation of a B-spline curve is shown here. We follow the notations of Piegl and Tiller [10] in this work.

Let $\boldsymbol{C}_p = \sum_{i=0}^{n} N_{i,p}(u)\boldsymbol{P}_i$ be an end point interpolating (nonperiodic) degree p B-spline curve with respect to the knot vector U. To elevate its degree to $p+1$, one needs to construct a knot vector $\hat{U}$ and control points $\boldsymbol{Q}_i$ so that

$$\boldsymbol{C}_p(u) = \boldsymbol{C}_{p+1}(u) = \sum_{i=0}^{\hat{n}} N_{i,p+1}(u)\boldsymbol{Q}_i. \tag{9}$$

Assume U has the following form:

$$\begin{aligned} U &= \{u_0, \cdots, u_m\} \\ &= \{a, \cdots, a, u_1, \cdots, u_1, \cdots, \\ &\qquad u_s, \cdots, u_s, b, \cdots, b\} \end{aligned} \tag{10}$$

where the multiplicities of the interior knots are $m_1, \cdots, m_s$, respectively. At a knot of multiplicity m_i, $\boldsymbol{C}_p(u)$ is C^{p-m_i} continuous and $C_{p+1}(u)$ must have the same degree of continuity there. Therefore, $\hat{n} = n+s+1$ and

$$\begin{aligned} \hat{U} &= \{u_0, \cdots, u_{\hat{m}}\} \\ &= \{a, \cdots, a, u_1, \cdots, u_1, \cdots, \\ &\qquad u_s, \cdots, u_s, b, \cdots, b\} \end{aligned} \tag{11}$$

where $\hat{m} = m+s+2$. The control points $\boldsymbol{Q}_i$ of the degree-elevated curve $C_{p+1}(u)$ are determined by solving the following system of linear equations:

$$\sum_{i=0}^{\hat{n}} N_{i,p+1}(u_j)\boldsymbol{Q}_i = \sum_{i=0}^{n} N_{i,p}(u_j)\boldsymbol{P}_i, \quad j = 0, \cdots, \hat{n}, \tag{12}$$

where u_j are $\hat{n}+1$ appropriate parameter values. The degree elevation of a NURBS curve can be done similarly and there are more efficient methods to calculate $\boldsymbol{Q}_i$ [10].

In case of a periodic curve, if the knot vector U is $\{u_0, u_1, \cdots, u_m\}$, then $\hat{U}$ is given by

$$\hat{U} = \{u_1, u_1, \cdots, u_{m-1}, u_{m-1}\}. \tag{13}$$

For example, the closed quadratic B-spline curve constructed from the triangular control polygon in Figure 1(a) consists of three segments and its knot vector U is as follows:

$$U = \{-\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}\}. \tag{14}$$

The knot vector $\hat{U}$ and the control points $\boldsymbol{Q}_i$ of the degree elevated cubic B-spline curve are given by

$$U = \{-\frac{1}{3}, -\frac{1}{3}, 0, 0, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, 1, 1, \frac{4}{3}, \frac{4}{3}\}, \tag{15}$$

and

$$\begin{aligned} \boldsymbol{Q}_{2i} &= \frac{5}{6}\boldsymbol{P}_i + \frac{1}{6}\boldsymbol{P}_{i+1}, \\ \boldsymbol{Q}_{2i+1} &= \frac{1}{6}\boldsymbol{P}_i + \frac{5}{6}\boldsymbol{P}_{i+1}, \end{aligned} \tag{16}$$

for $i = 0, \cdots, 3$. The original quadratic B-spline curve and the degree elevated curve can be regarded as a uniform and a non-uniform B-spline subdivision curve, respectively. The knot intervals of the degree elevated curve are shown in Figure 1(d).

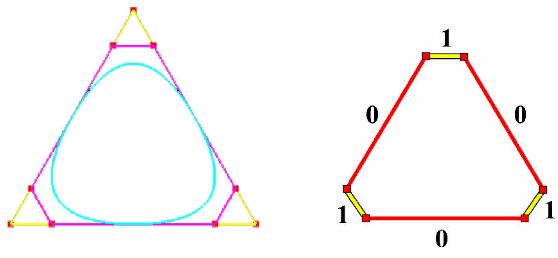
3.2 Degree elevation of subdivision curves

Chaikin's algorithm [2] generates a quadratic B-spline curve from a control polygon through recursive subdivision. Each subdivision step generates two new points on each polygon leg. If there are $n+1$ vertices $\boldsymbol{p}_i^j$, $i = 0, ..., n$, after the j-th recursive subdivision, then the two new points generated for the polygon leg $\boldsymbol{p}_i^j\boldsymbol{p}_{i+1}^j$ are defined as follows:

$$\boldsymbol{p}_{2i}^{j+1} := \frac{3}{4}\boldsymbol{p}_i^j + \frac{1}{4}\boldsymbol{p}_{i+1}^j, \qquad (17)$$

$$\boldsymbol{p}_{2i+1}^{j+1} := \frac{1}{4}\boldsymbol{p}_i^j + \frac{3}{4}\boldsymbol{p}_{i+1}^j. \qquad (18)$$

The subdivision curve generated by Chaikin's algorithm is a quadratic B-spline curve.



(a) Control polygons (Quadratic: yellow; non-uniform cubic: magenta).

(b) Knot intervals for the non-uniform cubic B-spline subdivision curve.

Figure 1: Quadratic and cubic B-spline subdivision curves.

The conversion of the control polygon of a quadratic B-spline curve to that of a non-uniform cubic B-spline curve is accomplished by generating two new points defined as follows for each polygon leg $\boldsymbol{p}_i\boldsymbol{p}_{i+1}$,

$$\boldsymbol{q}_{2i} := \frac{5}{6}\boldsymbol{p}_i + \frac{1}{6}\boldsymbol{p}_{i+1}, \qquad (19)$$

$$\boldsymbol{q}_{2i+1} := \frac{1}{6}\boldsymbol{p}_i + \frac{5}{6}\boldsymbol{p}_{i+1}, \qquad (20)$$

in a manner similar to Chaikin's algorithm. The converted control polygon is the control polygon of a non-uniform cubic B-spline subdivision curve proposed by Sederberg et al. [12]. For a periodic cubic B-spline curve, there is an one-to-one correspondence between the edges of the control polygon and the segments of the curve. Following the same notation of [12], a knot interval between two consecutive knots is assigned to each control edge of the converted curve, as shown in 1(b).

Note that for a periodic quadratic B-spline curve, there is an one-to-one correspondence between the vertices of the control polygon and the segments of the curve. However, for a periodic linear B-spline curve the correspondence is between the edges of the control polygon and the segments.

The triangle in yellow in Figure 1(a) is a control polygon of a quadratic B-spline subdivision curve and the hexagon in pink in the same figure is that of a non-uniform cubic B-spline subdivision curve. The triangle is overlaped by the hexagon and some parts of the triangle is not visible. These control polygons generate an identical curve draw in blue. The knot intervals for the non-uniform cubic curve are shown in Figure 1(b).

4 Conversion from the Doo-Sabin to Catmull-Clark subdivision surfaces

The approximate conversion from a Doo-Sabin subdivision surface to a non-uniform Catmull-Clark subdivision surface can be performed topologically in the same way as the Doo-Sabin subdivision, but geometrically they are different and different subdivision coefficients are used for the conversion process. We apply to the Doo-Sabin surface a linear B-spline function whose parameter domain is identical to that of the original surface.

For a regular mesh, as the one shown in Figure 3(a), the conversion may be regarded as a degree elevation of a uniform biquadratic surface to a non-uniform bicubic B-spline surface. In the figure, the green lines are patch boundaries of the quadratic B-spline surface. The yellow and red lines are those of the non-uniform cubic B-spline surface whose knot intervals are 1 and 0, respectively. The new control points $\boldsymbol{F}_a$ is given by

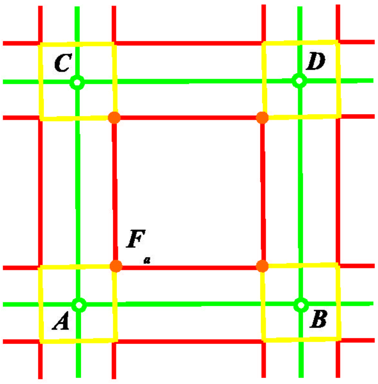
$$\begin{aligned}\boldsymbol{F}_a &= \frac{25}{36}\boldsymbol{A} + \frac{5}{36}(\boldsymbol{B}+\boldsymbol{C}) + \frac{1}{36}\boldsymbol{D} \\ &= \frac{(\boldsymbol{V}+2\boldsymbol{A})}{3} + \frac{\boldsymbol{B}+\boldsymbol{C}-\boldsymbol{A}-\boldsymbol{D}}{9}, \qquad (21)\end{aligned}$$

where $\boldsymbol{V} = (\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{C}+\boldsymbol{D})/4$.

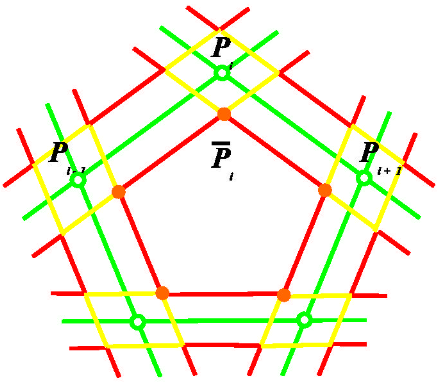
The n-sided faces where $n \neq 4$ remain n-sided after the conversion in a manner identical to the Doo-Sabin subdivision. In Figure 3(b), the new vertex $\bar{\boldsymbol{P}}_i$ is calculated by

$$\begin{aligned}\bar{\boldsymbol{P}}_i &= (\frac{4}{9} + \frac{1}{n})\boldsymbol{P}_i \\ &\quad \frac{1}{9n} + \sum_{j=1, j\neq i}^{n} (5 + 4\cos(\frac{2\pi|i-j|}{n}))\boldsymbol{P}_{\boldsymbol{j}}. \qquad (22)\end{aligned}$$

The yellow and red lines are boundaries of the non-uniform cubic B-spline surface whose knot intervals are 1 and 0, respectively, as the regular mesh case.



(a) Regular mesh.



(b) Non-four-sided face.

Figure 2: **The conversion process.**

For a regular mesh, the above conversion is exact in the sense that a Doo-Sabin subdivision surface is exactly converted to a non-uniform Catmull-Clark subdivision surface. However, for an irregular mesh, the conversion is approximate. For example, the point $\boldsymbol{P}_0$ of the Doo-Sabin control mesh in Figure 3(c) will converge to $\boldsymbol{P}_0^\infty$:

$$\boldsymbol{P}_0^\infty = \frac{9}{16}\boldsymbol{P}_0 + \frac{1}{8}(\boldsymbol{P}_1 + \boldsymbol{P}_2 + \boldsymbol{P}_3) + \frac{1}{48}(\boldsymbol{P}_4 + \boldsymbol{P}_5 + \boldsymbol{P}_6). \quad (23)$$

The corresponding point $\boldsymbol{Q}_0^\infty$ of the converted Catmull-Clark surface is given by

$$\boldsymbol{Q}_0^\infty = \frac{629}{1152}\boldsymbol{P}_0 + \frac{223}{172}(\boldsymbol{P}_1 + \boldsymbol{P}_2 + \boldsymbol{P}_3) + \frac{77}{3456}(\boldsymbol{P}_4 + \boldsymbol{P}_5 + \boldsymbol{P}_6). \quad (24)$$

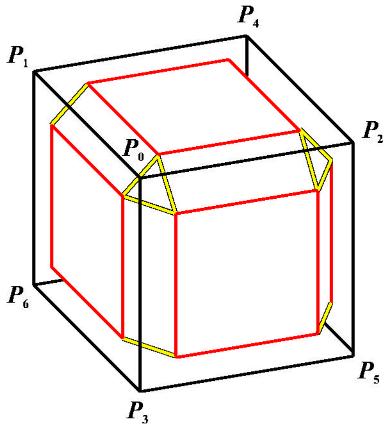


Figure 3: **Conversion error.**

The difference of these two points is

$$\boldsymbol{P}_0^\infty - \boldsymbol{Q}_0^\infty \approx 0.0165\boldsymbol{P}_0 - 0.00405(\boldsymbol{P}_1 + \boldsymbol{P}_2 + \boldsymbol{P}_3) - 0.00145(\boldsymbol{P}_4 + \boldsymbol{P}_5 + \boldsymbol{P}_6). \quad (25)$$

This is small enough for practical CG applications where extremely high degree is not the primary concern. The primary concern is the deformation process.

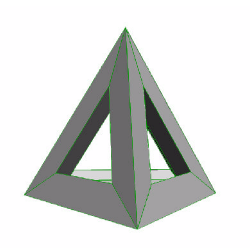
5 Conversion examples

Figure 4(c) shows a typical Doo-Sabin surface defined by the control mesh shown in 4(a). The control mesh is converted to the control mesh of a non-uniform Catmull-Clark surface in 4(b) where 1 is assigned as the knot interval to each edge in yellow color and 0 to each edge in red. Figure 4(d) shows a converted non-uniform Catmull-Clark surface

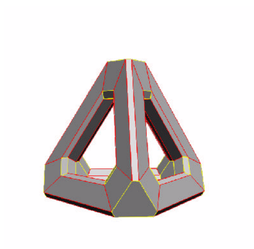
Figure 4(g) shows another example of the Doo-Sabin subdivision surface and its control mesh is shown in 4(e). The control mesh of the converted non-uniform Catmull-Clark surface and itself are shown in 4(f) and (g).

6 Conclusions

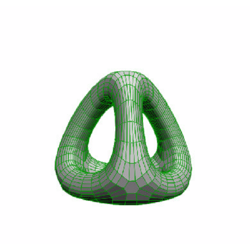
We have discussed the relationship between the Doo-Sabin and Catmull-Clark surfaces and have shown how to convert the Doo-Sabin to Catmull-Clark surfaces by elevating the degree of the original surface by one. In our method, the usual uniform Doo-Sabin surface can be approximately converted to a special type of the NURSS: a non-uniform Catmull-Clark surface.



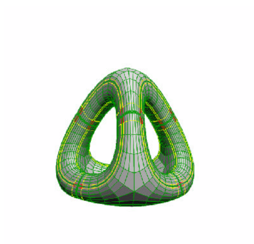
(a) Original control mesh.



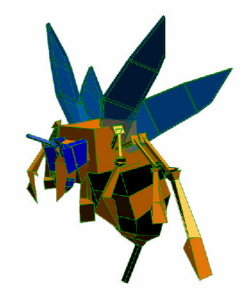
(b) Converted control mesh.



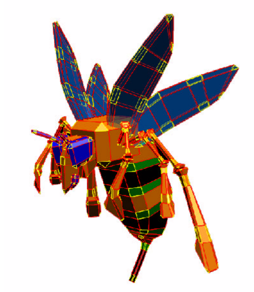
(c) Doo-Sabin surface (depth=3).



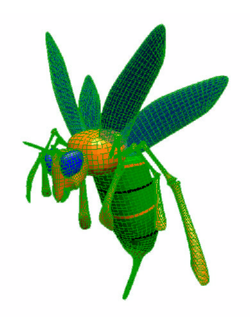
(d) Non-uniform Catmull-Clark surface (depth=3).



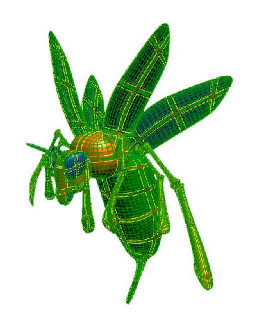
(e) Original control mesh.



(f) Converted mesh.



(g) Doo-Sabin surface (depth=3).



(h) Non-uniform Catmull-Clark surface (depth=3).

Figure 4: Examples of converted surfaces.

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